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The Sine Table of Al-Kashi in His Miftāḥ al-Ḥisab

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ABSTRACT
This article examines Al-Kāshī's Khāqānī sine table in Miftāḥ al-Ḥisab, converting it from the sexagesimal to decimal system. Our investigation includes a comparative analysis with a modern sine table to evaluate the accuracy of the Khāqānī sine values. Additionally, we address a copyist error in the table, providing corrections and subsequently restoring the table.

Keywords: Al-Kāshī, classical mathematics, Islamic mathematics history, Khāqānī sine table.

ABSTRAK
Dalam artikel ini, kami mengkaji jadual sin Khāqānī oleh Al-Kāshī dalam Miftāḥ al-Ḥisab. Kami menukarkan jadual dari sistem seksagesimal kepada sistem desimal dan menyiasat tahap ketepatan jadual sin Khāqānī dengan membandingkannya dengan jadual sin moden. Kami juga membuat pembetulan terhadap kesalahan yang telah dilakukan oleh penyalin dan seterusnya memulihkan kembali jadual tersebut.

Kata Kunci: Al-Kāshī, matematik klasik, sejarah matematik Islam, Khāqānī sine table.

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1. Introduction

In the year 1427, Ghiyath al-Din Jamshid bin Mas'ud bin Mahmood al-Tabib al-Kashi completed a book titled Miftah al-Hisab. This book was dedicated to Sultan Ulugh Beg in Samarkand and has been regarded as the crown of Islamic arithmetic brilliance. According to Berggren, it stands as the most fitting gift truly for a king. This book, originally penned in Arabic, saw a significant milestone in 2019 when the Miftah was successfully translated into English by Nuh Aydin, Lakhdar Hammoudi, and Ghada Bakbouk. The translation of the Miftah has been divided into three volumes, each dedicated to one of the three principal topics: arithmetic, geometry, and algebra. Having reading the translated Miftah, I found myself captivated by the sine table, a table for computing the angle between the base and the hypothenuse of a right triangle, compiled by al-Kashi in his Miftah. The table however is written in sexagesimal system, a numeral system of base 60. We can observe that al-Kashi has compiled a table of sine values ranging from 0 degrees to 90 degrees, precisely up to three sexagesimal places. This spurred a pondering within me: does the sine table articulated by al-Kashi in his work align with the contemporary sine tables utilized in our present day? Given that the table was constructed using the sexagesimal system, in order to understand the al-Kashi sine table, I have converted the table to the decimal system, a numeral system base 10, that we are familiar with.

What is surprising is that I have encountered several errors within the table. However, I contend that such errors should not be attributed to a brilliant mathematician like al-Kashi, given his title as a living calculator. I am of the opinion that these errors were introduced by the scribe or copyist of the book. The copyist inadvertently made typographical mistakes, raising the question of how we can discern whether these errors stem from the copyist or are indeed errors made by al-Kashi himself. Interestingly, al-Kashi himself acknowledged in his Miftah that it is customary for copyists to make errors in their reproductions, as I quoted what has been mentioned by al-Kashi himself in his book,

"*So that if there is a mistake in copying them from one place to another, it will be easy to correct due to the connection between them.*"

Although al-Kashi did not specifically mention about the sine table, his words apply to the entirety of his text.

In addressing this issue, al-Kashi demonstrated remarkable foresight. I have discovered that in al-Kashi's sine table, there is a column of values which are not only serves as elements in the table but also functions as an error-detecting and error-correcting code simultaneously. Utilizing these values, I have successfully reconstructed the original table authored by al-Kashi. Therefore, I will unveil all of my findings in the following sections.

Section 2 provides a comprehensive overview of Al-Kashi, offering a glimpse into the life and accomplishments of his mathematical luminary. Section 3, a focal point of the analysis, takes the reader through the meticulous process of converting Al-Kashi's original sine table into the decimal system, facilitating a more accessible understanding and comparison with contemporary mathematical frameworks.

In Section 4, the author embark on an exploration of the manuscript's nuances, unraveling typographical error in the sine table.

2. Early Life of Al-Kāshī.

Born in the city of Kashan in 1380, the renowned Persian mathematician and astronomer, Ghiyath al-Din Jamshid

Mas'ud al-Kashi, made significant contributions to the fields of mathematics and astronomy during the Islamic Golden Age. Al-Kashi's life spanned a pivotal period in the history of the Islamic world, marked by intellectual flourishing and cultural exchange.

Al-Kashi's intellectual prowess became evident early in his life, and he dedicated himself to the pursuit of knowledge, particularly in mathematics and astronomy. His seminal work, "Miftah al-Hisab," completed on March 2, 1427, stands as a testament to his mathematical acumen. This comprehensive treatise, meaning "The Key to Arithmetic," showcased al-Kashi's mastery of various mathematical concepts, including algebra and arithmetic. Notably, he dedicated this work to Ulugh Beg, the ruler of Samarkand, highlighting the recognition and patronage al-Kashi received from the influential figures of his time.

In addition to "Miftah al-Hisab," al-Kashi authored several other notable works. Around the years 1413-1414, he penned the "Khaqani Zij," a significant astronomical handbook that contributed to the understanding of celestial movements. His "Sullam al-Sama," or "The Stairway to the Heavens," further delved into celestial matters, providing insights into the structure of the cosmos.

One of al-Kashi's remarkable achievements was the composition of Risālah al-muḥīṭiyyah or "The Treatise on the Circumference," where he made noteworthy advancements in approximating the value of π (pi). His meticulous calculations and innovative approaches demonstrated his commitment to pushing the boundaries of mathematical knowledge.

Al-Kashi's legacy extends beyond his intellectual contributions. He played a pivotal role in preserving and transmitting the scientific knowledge of ancient civilizations, contributing to the rich tapestry of Islamic scholarship. His works continued to influence mathematicians and astronomers for centuries, leaving an indelible mark on the history of science in the Islamic world and beyond.

The book, aptly titled "Miftah al-Hisab," reflects its contents— a rich compilation encompassing algebra, geometry, and arithmetic. It goes beyond mere computation, extending to practical applications such as the determination of areas and volumes for various geometric shapes. Al-Kashi's motivation was clear: to provide engineers with a valuable resource for swift and accurate mathematical calculations, enhancing their efficiency in practical applications.

Noteworthy is al-Kashi's commitment to constructing original tables within "Miftah al-Hisab." While acknowledging the utility of tables for engineers, he underscored the uniqueness of his contributions. One standout table, the "Khaqani Sine Table," marked a special distinction within the treatise. This table, devoted to trigonometric calculations, showcased al-Kashi's depth of insight into mathematical principles related to calculating the angle between the base and the hypothenuse of a right triangle.

Al-Kashi has left us a rich legacy of contributions to mathematics, for which we hold deep gratitude. He passed away in the city of Samarkand on June 22, 1429. Despite various speculations surrounding the circumstances of his death, we continue to honor him as the Ptolemy the Second, acknowledging his profound impact on the field of mathematics and beyond.

3. Converting the Table from Sexagesimal into Decimals System.

The conversion of the Khaqani table from the sexagesimal system to the decimal system involves employing the Abjad numeral system—a numerical notation

system utilized by Islamic astronomers for representing numbers. In this unique system, each letter of the Arabic alphabet is assigned a specific numerical value. The Abjad values follow an ordered sequence from 1 to 9, then 10 to 90, and finally 100 to 1000 as we illustrated in the Table 1.

1	2	3	4	5	6	7	8	9	0
ا	ب	ج	د	ه	و	ز	ح	ط	ء

10	20	30	40	50	60	70	80	90
ي	ك	ل	م	ن	س	ع	ف	ص

100	200	300	400	500	600	700	800	900	1000
ق	ر	ش	ت	ث	خ	ذ	ض	ظ	غ

Table 1 The conversion table for Abjad numerical values.

The original sine table in Miftah was transcribed by using the Abjad numeral system as illustrated in the Table 1. This table serves as a key reference for seamlessly translating numerical values from the Abjad system into their corresponding Arabic counterparts. In essence, it provides a bridge between the unique symbolic representations of the Abjad system and the more familiar Arabic numeral system, facilitating a clearer understanding and utilization of the sine values encoded in Miftah.

al-Tafādhol التفاضل (Difference)		al-Jayb الجيب (Sine)			al-Qaus القوس (Arc)		al-Tafādhol التفاضل (Difference)		al-Jayb الجيب (Sine)			al-Qaus القوس (Arc)	
س	ق	س	ق	د			س	ق	س	ق	د		
ـه	مج	لز	ط	مج	مو	46	ن	سب	ء	ء	ء	ء	0
كز	مب	نب	نب	مج	مز	47	مح	سب	ن	ب	ا	ا	1
لح	ما	بط	له	مد	مح	47	مز	سب	لح	ه	ب	ب	2
مط	م	نز	بو	مه	مط	49	مب	سب	كه	ح	ج	ج	3
نح	لط	مو	نز	مه	ن	50	كط	سب	ز	ـا	د	د	4
و	لط	مد	لز	مو	نا	51	لب	سب	مو	لج	ه	ه	5
ـه	لح	ن	بو	مز	نب	52	كو	سب	بح	بو	و	و	6
كج	لز	ه	نه	مز	نج	53	نز	سب	مد	بح	ز	ز	7
كط	لو	كح	لب	مح	ند	54	ط	سب	ا	كا	ح	ح	8
كه	كه	نز	ح	مط	نه	55	نح	سا	ى	كج	ط	ط	9
ما	لد	لب	مد	مط	نو	56	مز	سا	ح	كه	ى	ى	10
مه	لج	بج	بط	ن	نز	57	لد	سا	نه	كو	ـا	ـا	11
ن	كب	نح	نب	ن	نح	58	ك	سا	كط	كح	ـب	ـب	12
نب	كا	مح	كه	نا	نط	59	و	سا	مط	كط	بج	بج	13
نز	ل	ما	نز	نا	س	60	ن	س	نه	ل	ند	ند	14
نط	كط	لح	كح	نب	سا	61	س	لب	مه	لا	ـه	ـه	15
ء	كط	لز	نح	نب	سب	62	س	ـه	بز	لب	بو	بو	16
ج	كح	لز	كز	نج	سج	63	نو	نط	لب	لب	بز	بز	17
ب	كز	م	نه	نج	سد	64	كه	نط	كح	لب	بح	بح	18
د	كو	مب	كب	ند	سه	65	بج	نط	ج	لب	نط	نط	19
ج	كه	مو	مح	ند	سو	66	نا	نح	لا	بو	ك	ك	20
ج	كد	مط	بج	نه	سز	67	كح	نح	ز	ل	كا	كا	21
ا	كج	نب	لز	نه	سح	68	ج	نح	له	كج	كب	كب	22
ا	كب	نج	ء	نو	سط	69	لز	نز	لح	كو	كج	كج	23
نح	ك	ند	كب	نو	ع	70	ـا	نز	ـه	كد	كد	كد	24
نو	نط	نب	مج	نو	عا	71	نب	نو	كو	كا	كه	كه	25
ند	لح	مح	ج	نز	عب	72	نو	ند	ح	بح	كو	كو	26
نا	بز	مب	كب	نز	عج	73	مد	نه	كب	ند	كز	كز	27
مز	بو	لج	م	نز	عد	74	بج	نه	و	ى	كح	كح	28
مد	ـه	ك	نز	نز	عه	75	ما	ند	نط	ه	كط	كط	29
م	ند	د	بج	نح	عو	76	ح	ند	ء	ء	ل	ل	30
لو	بج	مد	كز	نح	عز	77	له	نج	ح	ند	ل	لا	31
لا	بب	ك	ما	نح	عح	78	نط	نب	مب	مز	لا	لب	32
كز	ـا	نا	نج	نح	عط	79	نب	كد	مب	م	لب	لج	33
كج	ى	لح	ه	نط	ف	80	مز	نا	و	لج	لج	لد	34
بز	ط	ما	ـه	نط	فا	81	ط	نا	نج	كد	لد	له	35
بب	ح	نح	كد	نط	فب	82	ك	ن	ب	بو	له	لو	36
ز	ز	ى	لج	نط	فج	83	نا	مط	لب	و	لو	لز	37
ا	و	بز	م	نط	فد	84	ى	مط	كج	نو	لو	لح	38
نو	د	لح	مو	نط	فه	85	مح	كح	لج	مه	لز	لط	39
ن	ج	نا	بد	نط	فو	86	مز	مز	ب	لد	لح	م	40
مد	ب	د	نه	نط	فز	87	ج	مز	مط	كا	لط	ما	41
لط	ا	مح	نز	نط	فح	88	ك	مو	نب	ح	م	مب	42
لج	ء	كز	نط	نط	فط	89	لد	مه	ب	نه	م	مج	43
		ء	ء	س	ص	90		مد	مو	م	ما	مد	44
							ب	مد	له	كه	مب	مه	45

Khaqani sine table derived from the Miftah.

In the original Khaqani table, there is uncertainty regarding the character that Al-Kashi employed to represent zero. To address this ambiguity, a pragmatic decision has been made to designate the character hamzah, ء as a placeholder for zero. This choice is based on the visual resemblance of hamzah to a potential representation of zero. It is important to note that this substitution is a practical measure taken due to the lack of explicit information on the zero character used by Al-Kashi in the Khaqani table.

Based on the aformentioned Khaqani sine table, we have converted Table 2 from the sexagesimal system to the

decimal system, as demonstrated by Tables 3 and 4 respectively. Remarkably, subsequent to the conversion process, a striking resemblance between the Khaqani sine table and its modern counterpart has been observed. However, upon meticulous scrutiny and a comparative analysis with the contemporary table (Royster, 2017) discernible discrepancy has surfaced, particularly in the value corresponding to the arc of 50. These identified error merit closer examination and consideration in the context of refining the accuracy of the converted sine table.

4. Error Detecting and Corrections.

In the translation of the Miftah by Nuh Aydin and Hammoudi (2020), a mistake was identified regarding the arc of 50 degrees. The error arose during the conversion of the Abjad numerals to Arabic numerals, where the arc 50 was incorrectly transliterated as 50:45:57:46 with a calculated difference of 29:58. However, upon examination of the original table, it becomes evident that the correct numeral should have been 39 (notated as لط in the original), not 29. This discrepancy led to an incorrect calculation of the difference (al-Tafãdhol) as 0.0083, diverging from the accurate difference between 0.7771 and 0.7660, which is 0.0111. By amending the numeral from 29 to 39, the corrected difference (al-Tafãdhol) aligns with the expected ratio of 0.0111, indicating that the error originated from the translation process. Therefore, it is established that the accurate value is 39, not 29, pinpointing a translation error in the arc of 50 degrees.

6. Conclusion

In the aforementioned discussion, it was established that the Khaqani sine table, devised by al-Kashi, contains no errors, with identified discrepancies arising from the translator's misinterpretation during the transition from Abjad to Arabic numerals. While Hamadanizadeh (1980) had previously undertaken the task of converting the Khaqani sine table from sexagesimal to decimal notation, his work incorporated only a minor segment of this extensive table. Our research, however, extends this effort by converting the full table, ranging from 0 to 90 degrees. Remarkably, we observed that al-Kashi's Khaqani sine table aligns flawlessly with modern sine tables and surpasses them in accuracy, offering precision up to eight decimal places as opposed to the four decimal places typical of contemporary tables, a refinement attributable to the inclusion of second values in its calculations. This revelation highlights the advanced mathematical knowledge within the Islamic world by the 15th century, showcasing their early adoption and widespread use of remarkably precise sine tables.

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al-Qaus القوس (Arc)		al-Jayb الجيب (Sine)			al-Tafādhol التفاضل (Difference)		al-Jayb الجيب (Sine)	al-Tafādhol التفاضل (Difference)	al-Jayb الجيب (Sine)	al-Tafādhol التفاضل (Difference)
		D	M	S	M	S	In decimal	In decimal	In ratio	In ratio
0		0	0	0	62	50	0	1.0472	0	0.0175
1		1	2	50	62	48	1.0472	1.0467	0.0175	0.0174
2		2	5	38	62	47	2.0939	1.0464	0.0349	0.0174
3		3	8	25	62	42	3.1403	1.0450	0.0523	0.0174
4		4	11	7	62	29	4.1853	1.0414	0.0698	0.0174
5		5	13	46	62	32	5.2294	1.0422	0.0872	0.0174
6		6	16	18	62	26	6.2717	1.0456	0.1045	0.0174
7		7	18	44	62	17	7.3122	1.0381	0.1219	0.0173
8		8	21	1	62	9	8.3503	1.0358	0.1392	0.0173
9		9	23	10	61	58	9.3861	1.0328	0.1564	0.0172
10		10	25	8	61	47	10.4189	1.0297	0.1736	0.0172
11		11	26	55	61	34	11.4486	1.0261	0.1908	0.0171
12		12	28	29	61	20	12.4747	1.0222	0.2079	0.0170
13		13	29	49	61	6	13.4969	1.0183	0.2249	0.0170
14		14	30	55	60	50	14.5153	1.0139	0.2419	0.0169
15		15	31	45	60	32	15.5292	1.0089	0.2588	0.0168
16		16	32	17	60	15	16.5381	1.0042	0.2756	0.0167
17		17	32	32	59	56	17.5422	0.9989	0.2924	0.0166
18		18	32	28	59	25	18.5411	0.9903	0.3090	0.0165
19		19	32	3	59	13	19.5342	0.9869	0.3256	0.0164
20		20	31	16	58	51	20.5211	0.9808	0.3420	0.0163
21		21	30	7	58	28	21.5019	0.9744	0.3584	0.0162
22		22	28	35	58	3	22.4764	0.9675	0.3746	0.0161
23		23	26	38	57	37	23.4439	0.9603	0.3907	0.0160
24		24	24	15	57	11	24.4042	0.9531	0.4067	0.0159
25		25	21	26	56	52	25.3572	0.9478	0.4226	0.0158
26		26	18	8	56	14	26.3022	0.9372	0.4384	0.0156
27		27	14	22	55	44	27.2394	0.9289	0.4540	0.0155
28		28	10	6	55	13	28.1683	0.9203	0.4695	0.0153
29		29	5	19	54	41	29.0886	0.9114	0.4848	0.0152
30		30	0	0	54	8	30.0000	0.9022	0.5000	0.0150
31		30	54	8	53	35	30.9022	0.8931	0.5150	0.0149
32		31	47	42	52	59	31.7950	0.8832	0.5299	0.0147
33		32	40	42	52	24	32.6783	0.8733	0.5446	0.0146
34		33	33	6	51	47	33.5517	0.8631	0.5592	0.0144
35		34	24	53	51	9	34.4147	0.8525	0.5736	0.0142
36		35	16	2	50	20	35.2672	0.8389	0.5878	0.0140
37		36	6	32	49	51	36.1089	0.8308	0.6018	0.0138
38		36	56	23	49	10	36.9397	0.8194	0.6157	0.0137
39		37	45	33	48	28	37.7592	0.8078	0.6293	0.0135
40		38	34	2	47	47	38.5672	0.7964	0.6428	0.0133
41		39	21	49	47	3	39.3636	0.7842	0.6561	0.0131
42		40	8	52	46	20	40.1478	0.7722	0.6691	0.0129
43		40	55	12	45	34	40.9200	0.7594	0.6820	0.0127
44		41	40	46	44	49	41.6794	0.7469	0.6947	0.0124
45		42	25	35	44	2	42.4264	0.7339	0.7071	0.0122
46		43	9	37	43	15	43.1603	0.7208	0.7193	0.0120
47		43	52	52	42	27	43.8811	0.7075	0.7314	0.0118
48		44	35	19	41	38	44.5886	0.6939	0.7431	0.0116
49		45	16	57	40	49	45.2825	0.6803	0.7547	0.0113
50		45	57	46	29/39	58	45.9628	0.4994/0.6661	0.7660	0.0083/0.0111

Table 3 Khaqani sine table for degree 0 to 50.

al-Qaus القوس (Arc)		al-Jayb الجيب (Sine)			al-Tafādhol التفاضل (Difference)		al-Jayb الجيب (Sine)	al-Tafādhol التفاضل (Difference)	al-Jayb الجيب (Sine)	al-Tafādhol التفاضل (Difference)
		D	M	S	M	S	In decimal	In decimal	In ratio	In ratio
51		46	37	44	39	6	46.6289	0.6517	0.7771	0.0109
52		47	16	50	38	15	47.2806	0.6375	0.7880	0.0106
53		47	55	5	37	23	47.9181	0.6231	0.7986	0.0104
54		48	32	28	36	29	48.5411	0.6081	0.8090	0.0101
55		49	8	57	25	25	49.1492	0.4236	0.8192	0.0071
56		49	44	32	34	41	49.7422	0.5781	0.8290	0.0096
57		50	19	13	33	45	50.3203	0.5625	0.8387	0.0094
58		50	52	58	22	50	50.8828	0.3806	0.8480	0.0063
59		51	25	48	21	52	51.4300	0.3644	0.8572	0.0061
60		51	57	41	30	57	51.9614	0.5158	0.8660	0.0086
61		52	28	38	29	59	52.4772	0.4997	0.8746	0.0083
62		52	58	37	29	0	52.9769	0.4833	0.8829	0.0081
63		53	27	37	28	3	53.4603	0.4675	0.8910	0.0078
64		53	55	40	27	2	53.9278	0.4506	0.8988	0.0075
65		54	22	42	26	4	54.3783	0.4344	0.9063	0.0072
66		54	48	46	25	3	54.8128	0.4175	0.9135	0.0070
67		55	13	49	24	3	55.2303	0.4008	0.9205	0.0067
68		55	37	52	23	1	55.6311	0.3836	0.9272	0.0064
69		56	0	53	22	1	56.0147	0.3669	0.9336	0.0061
70		56	22	54	20	58	56.3817	0.3494	0.9397	0.0058
71		56	43	52	19	56	56.7311	0.3322	0.9455	0.0055
72		57	3	48	18	54	57.0633	0.3150	0.9511	0.0053
73		57	22	42	17	51	57.3783	0.2975	0.9563	0.0050
74		57	40	33	16	47	57.6758	0.2797	0.9613	0.0160
75		57	57	20	15	44	57.9556	0.2622	0.9659	0.0044
76		58	13	4	14	40	58.2178	0.2444	0.9703	0.0041
77		58	27	44	13	36	58.4622	0.2267	0.9744	0.0038
78		58	41	20	12	31	58.6889	0.2086	0.9781	0.0035
79		58	53	51	11	27	58.8975	0.1908	0.9816	0.0032
80		59	5	18	10	23	59.0883	0.1731	0.9848	0.0029
81		59	15	41	9	17	59.2614	0.1547	0.9877	0.0026
82		59	24	58	8	12	59.4161	0.1367	0.9903	0.0023
83		59	33	10	7	7	59.5528	0.1186	0.9925	0.0020
84		59	40	17	6	1	59.6714	0.1003	0.9945	0.0017
85		59	46	18	4	56	59.7717	0.0822	0.9962	0.0014
86		59	51	14	3	50	59.8539	0.0639	0.9976	0.0011
87		59	55	4	2	44	59.9178	0.0456	0.9986	0.0008
88		59	57	48	1	39	59.9633	0.0275	0.9994	0.0005
89		59	59	27	0	33	59.9908	0.0092	0.9998	0.0002
90		60	0	0			60.0000		1.0000	

Table 4 Khaqani sine table for degree 51 to 90.

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