Analytical Solution of The Two-Qubit Quantum Rabi Model

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Abstract− In this paper, an analytical solution of the two-qubit Rabi model for the general case is presented. Furthermore, a comparison between the information entropies and the Von Neumann entropy $S(\rho_A)$ is given for some special values of the qubit-photon coupling constants for the two qubits and the detuning parameters. It is demonstrated that oscillations of the occupation probabilities ρ_{11}, ρ_{22}, ρ_{33} and ρ_{44} are equivalent to the case of the spontaneous emission. The occupation probability ρ_{11} reaches the case of sudden death, when the detuning parameter Δ_2 equals zero.

Keywords− Quantum Optics and Rabi Model.

I. Introduction

The light-matter interaction is one of the fundamental problems investigated in many aspects of modern physics ranging from quantum optics to quantum information processing and to condensed matter physics. The simple description of quantum light-matter interaction, namely the interaction of a qubit with a harmonic oscillator, is given by Rabi model [1]. This simple model has been widely used in atomic physics, optical physics and condensed matter physics [2, 3]. In addition, it has been experimentally tested in different physical systems, such as cavity systems [4], trapped ions [5], quantum dots [6], superconducting circuits [7–12] and photonic superlattices [13,14]. Finding solutions of the quantum optical Hamiltonian has been an important topic in the existing literature [15–19]. Rabi model in the symmetric case has been studied by using Fulton-Gouterman transformation [20], in which solutions of this model are proposed as a function of energy [21–28]. Most of these solutions are concentrated the model calculations mainly on the G function. However, other attempts considered Rabi model solution as a function of time. However, such attempts used either rotating wave approximation (RWA), which leads to Jaynes-Cummings model [29, 30], or generalized rotating-wave approximation (GRWA) of only one qubit [31]. In this work, we focus on the analytical solution of the two-qubit Rabi model for the general case. By calculating the occupation probabilities of the the multi-qubit Rabi model, we show that it is possible to generate oscillations to a great extent look like the case of the spontaneous emission in different situations. In section II we describe the Hamiltonian of the given system, and obtain the explicit analytical solution of the model describing the dynamics of multi-qubit Rabi model. In Section III we discuss the Von Neumann entropy $S(\rho_A)$ and the information entropies $H(\sigma_Z)$, $H(\sigma_Y)$ and $H(\sigma_X)$ by changing the qubit-photon coupling constants for the two qubits and the detuning parameters.

II. Exact Solution of The Model

We briefly discuss the general formalism to characterize the dynamics of the multi-qubit Rabi model for the general case, for more details, see [32]. The Hamiltonian of the system can be written as ($\hbar = 1$):

$$H = \omega a^\dagger a + (a^\dagger + a) \sum_{j=1}^{2} \left(\mu_j \sigma_{j,x} + \delta_j \sigma_{j,z}\right), \qquad (1)$$

where $a^\dagger$ and a are the single-mode photon creation and annihilation operators with frequency ω, respectively. $\sigma_{ix}, \sigma_{iz} (i = 1, 2)$ are the Pauli matrices of the i^{th} qubit. δ_j is the detuning parameter of the qubit j, and μ_j is the qubit-photon coupling constants.

Here we start with the two-qubit case and try to find an analytic solution

$$\begin{aligned} H &= \omega a^\dagger a + \mu_1 \sigma_{1x}(a^\dagger + a) + \mu_2 \sigma_{2x}(a^\dagger + a) \\ &\quad + \delta_1 \sigma_{1z} + \delta_2 \sigma_{2z} \\ &= \omega\{a^\dagger a + g_1 \sigma_{1x} I_2 (a^\dagger + a) + g_2 I_1 \sigma_{2x}(a^\dagger + a) \\ &\quad + \Delta_1 \sigma_{1z} I_2 + \Delta_2 I_1 \sigma_{2z}\}, \end{aligned} \tag{2}$$

where $g_1 = \frac{\mu_1}{\omega}$, $g_2 = \frac{\mu_2}{\omega}$, $\Delta_1 = \frac{\delta_1}{\omega}$ and $\Delta_2 = \frac{\delta_2}{\omega}$. One may consider the following unitary transformations $S_1 = \frac{1}{\sqrt{2}}(\sigma_{1x} + \sigma_{1z})$ and $S_2 = \frac{1}{\sqrt{2}}(\sigma_{2x} + \sigma_{2z})$ to H to obtain $\grave{H}$ as

$$\begin{aligned} \grave{H} &= \omega\{a^\dagger a + g_1 \sigma_{1z}(a^\dagger + a) + g_2 \sigma_{2z}(a^\dagger + a) \\ &\quad + \Delta_1 \sigma_{1x} + \Delta_2 \sigma_{2x}\}, \end{aligned} \tag{3}$$

where $\grave{H} = U^\dagger H U$ and $U = S_2 \otimes S_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$, with $S_i = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}_i$.

Then, the general solution of the eigen value problem can be written as follows:

$$\begin{aligned} |\grave{\Psi}\rangle &= \sum_{k=1}^{4} \exp(-i\omega_k t)|\grave{\Psi}_k\rangle \\ &= \sum_{k=1}^{4} \exp(-i\lambda_k t) \begin{pmatrix} Q_{1k} \\ Q_{2k} \\ Q_{3k} \\ Q_{4k} \end{pmatrix} |\alpha\rangle, \end{aligned} \tag{4}$$

where
$\lambda_k = |\alpha|^2 + (-1)^k x_3$,
$Q_{1k} = \frac{y_1 y_2 \zeta_1 + (-1)^k \zeta_2 - x_1 x_3}{2\Delta_1 \Delta_2 x_3}$,
$Q_{2k} = (-1)^k \Big(\frac{y_1[y_1 + (-1)^k x_3] + \Delta_1^2 + (-1)^{\frac{k(k+1)}{2}} x_1}{2\Delta_1 x_3} \Big)$,
$Q_{3k} = (-1)^k \Big(\frac{y_2[y_2 + (-1)^k x_3] + \Delta_2^2 + (-1)^{\frac{k(k+1)}{2}} x_1}{2\Delta_2 x_3} \Big)$,
$Q_{4k} = \frac{1}{2},\ y_1 = (g + \grave{g})Re(\alpha),\ y_2 = (g - \grave{g})Re(\alpha)$,
$g = g_1 + g_2, \grave{g} = g_1 - g_2$,
$x_1 = \sqrt{(y_1^2 + \Delta_1^2)(y_2^2 + \Delta_2^2)}$,
$x_2 = \Delta_1^2 + \Delta_2^2 + 2[Re(\alpha)^2(g^2 + \grave{g}^2)]$
$x_3 = \sqrt{x_2 + (-1)^{\frac{k(k+1)}{2}} 2x_1}$,
$\zeta_1 = [(-1)^k 2gRe(\alpha) + x_3]$ and
$\zeta_2 = [y_2 \Delta_1^2 + y_1 \Delta_2^2 + (-1)^{\frac{k(k+1)}{2}} 2gRe(\alpha) x_1]$
However, from above system (4), one can note that $|\grave{\Psi}_i\rangle = \{K_1 |+,+,\alpha\rangle, K_2|+,-,\alpha\rangle, K_3|-,+,\alpha\rangle, K_4|-,-,\alpha\rangle\}$, with $|\alpha\rangle$ is the coherent state and $K_j = \sum_{k=1}^{4} \exp(-i\lambda_k t) Q_{jk}$. Then we obtain $|\Psi\rangle$ as follows $|\Psi\rangle = U|\grave{\Psi}\rangle$.

The density matrix operator of the whole system, $\hat{\rho}(t)$, can be written as $\hat{\rho}(t) = |\Psi\rangle|\Psi\rangle$. Hence, the density operator of the two qubits, $\hat{\rho}_R(t)$, is given by:

$$\hat{\rho}_R(t) = Tr_F(\hat{\rho}) = \frac{1}{4} V V^\dagger, \tag{5}$$

with $V = \begin{pmatrix} K_1 + K_2 + K_3 + K_4 \\ K_1 - K_2 + K_3 - K_4 \\ K_1 + K_2 - K_3 - K_4 \\ K_1 - K_2 - K_3 + K_4 \end{pmatrix}$.

In Fig. (1), we investigate the effect of the qubit-photon coupling constants for the two qubits (g_1 and g_2) on the atomic occupation probabilities ρ_{11}, ρ_{22}, ρ_{33} and ρ_{44}. The occupation probabilities ρ_{11}, ρ_{22}, ρ_{33} and ρ_{44} have regular and periodic oscillations. We show that when the qubit-photon coupling constants for the two qubits are equal ($g_1 = g_2$, $\grave{g} = 0$), the occupation probability ρ_{11} starts from its minimum value, $\rho_{11} = 0$, and it has maximum value at $\rho_{11} = 0.75$. While $g_1 = 3g_2$, $\grave{g} = 1$, the occupation probability ρ_{11} starts from the value, $\rho_{11} = 0.3$,and it has maximum value at $\rho_{11} = 0.5$. Also, when $g_1 = 7g_2$, $\grave{g} = 1.5$, the occupation probability ρ_{11} starts from the value, $\rho_{11} = 0.35$,and it has maximum value at $\rho_{11} = 0.5$, and the number of periodic oscillations increases. When the qubit-photon coupling constants for the two qubits are equal ($g_1 = g_2$, $\grave{g} = 0$), the occupation probability ρ_{44} starts from its maximum value, $\rho_{44} = 0.7$, and it has minimum value at $\rho_{44} = 0$. While $g_1 = 3g_2$, $\grave{g} = 1$, the occupation probability ρ_{44} starts from the value, $\rho_{44} = 0.3$,and it has maximum value at $\rho_{44} = 0.5$. Also, when $g_1 = 7g_2$, $\grave{g} = 1.5$, the occupation probability ρ_{44} starts from the value, $\rho_{44} = 0.35$,and it has maximum value at $\rho_{44} = 0.5$, and the number of periodic oscillations increases. So, we see that the second and the third cases of ρ_{11} and ρ_{44} are the same. When the qubit-photon coupling constants for the two qubits are equal ($g_1 = g_2$, $\grave{g} = 0$), the occupation probabilities ρ_{22} and ρ_{33} start from minimum value, $\rho_{22} = \rho_{33} = 0.1$, and they have small oscillations. While $g_1 = 3g_2$, $\grave{g} = 1$, the occupation probabilities ρ_{22} and ρ_{33} start from the value, $\rho_{22} = \rho_{33} = 0.15$,and the oscillations increase. Also, when $g_1 = 7g_2$, $\grave{g} = 1.5$, the occupation probabilities ρ_{22} and ρ_{33} start from the value, $\rho_{22} = \rho_{33} = 0.2$,and the oscillations increase.

In Fig. (2), we investigate the effect of the detuning parameters (Δ_1 and Δ_2) on the atomic occupation probabilities ρ_{11}, ρ_{22}, ρ_{33} and ρ_{44}. The occupation probabilities ρ_{11}, ρ_{22}, ρ_{33} and ρ_{44} have regular and periodic oscillations. We observe, when $\Delta_1 = 0.5$, $\Delta_2 = 0$, the oscillations of the occupation probabilities to a great extent look like the case of the spontaneous emission. The occupation probability ρ_{11} starts from its minimum value, $\rho_{11} = 0$, and increases until it

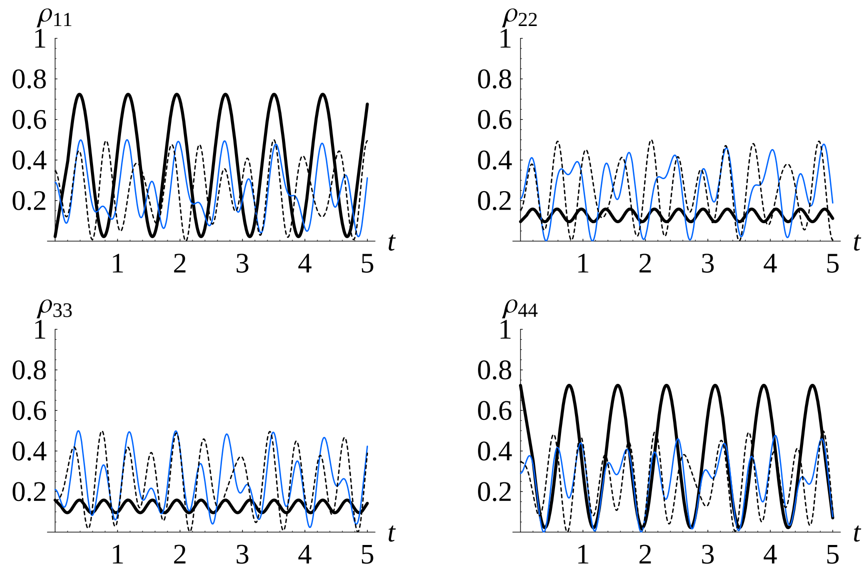

Figure 1: *These figures show the the cases in which $\omega = 1, \Delta_1 = \Delta_2 = 0.5, g = 2$, where solid, blue and dot curves correspond, respectively, to $\grave{g} = 0 (g_1 = g_2)$, $\grave{g} = 1$ $g_1 = 3g_2$ and $\grave{g} = 1.5$ $(g_1 = 7g_2)$.*

reaches maximum value at $\rho_{11} = 1$ then decreases and increases gradually until it reaches its minimum value, $\rho_{11} = 0$ (sudden death) and increases again and so on. While ($\Delta_1 = 0.5$, $\Delta_2 = 0.5$), the occupation probability ρ_{11} starts from the value, $\rho_{11} = 0.3$,and it has maximum value at $\rho_{11} = 0.5$. When ($\Delta_1 = 0.5$, $\Delta_2 = 0.8$), the occupation probability ρ_{11} starts from the value, $\rho_{11} = 0.35$,and it has maximum value at $\rho_{11} = 0.9$, and the number of periodic oscillations increase. The occupation probability ρ_{44} starts from its maximum value, $\rho_{11} = 1$, and decreases until it reaches minimum value at $\rho_{11} = 0$, then increases and decreases gradually until it reaches its minimum value, $\rho_{11} = 0$ (sudden death) and increases again and so on. While ($\Delta_1 = 0.5$, $\Delta_2 = 0.5$), the occupation probability ρ_{44} starts from the value, $\rho_{11} = 0.3$,and it has maximum value at $\rho_{44} = 0.5$. When ($\Delta_1 = 0.5$, $\Delta_2 = 0.8$), the occupation probability ρ_{44} starts from the value, $\rho_{44} = 0.35$,and it has maximum value at $\rho_{44} = 0.9$, and the number of periodic oscillations increases. The occupation probabilities ρ_{22} and ρ_{33} start from its minimum value, $\rho_{22} = \rho_{33} = 0$, and increase then decrease and increase gradually until they reach maximum value at $\rho_{22} = \rho_{33} = 1$ then decrease and increase gradually until it reaches the minimum value, $\rho_{22} = \rho_{33} = 0$ (sudden death) and increase again and so on. When ($\Delta_1 = 0.5$, $\Delta_2 = 0.5$), the occupation probabilities ρ_{22} and ρ_{33} start from the value, $\rho_{22} = \rho_{33} = 0.3$,and they have maximum value at $\rho_{22} = \rho_{33} = 0.5$. When ($\Delta_1 = 0.5$, $\Delta_2 = 0.8$), the occupation probabilities ρ_{22} and ρ_{33} start from the value, $\rho_{22} = \rho_{33} = 0.35$,and they have maximum value at $\rho_{22} = \rho_{33} = 0.9$, and the number of periodic oscillations increase. We observe, the case of ($\Delta_1 = 0.5$, $\Delta_2 = 0$) to a great extent look like the case of ($\Delta_1 = 0.5$, $\Delta_2 = 0.8$). But the case of ($\Delta_1 = 0.5$, $\Delta_2 = 0.5$) is different from the two former cases, where the difference between Δ_1 and Δ_2 is equal zero.

III. The Information Entropies

The probability distribution for N possible outcomes of measurements for an arbitrary quantum state of an operator σ_γ is

$$P(\sigma_\gamma) = \langle \Psi_{\gamma i} | \rho | \Psi_{\gamma i} \rangle , \quad (6)$$

where $|\Psi_{\gamma i}\rangle$ eigenvector of the operator σ_γ:

$$\sigma_\gamma |\Psi_{\gamma i}\rangle = \eta_{\gamma i} |\Psi_{\gamma i}\rangle , \gamma = x, y, z, i = 1, 2, ..., N. \quad (7)$$

For a two two-level atoms, $N = 4$ and by using the atomic reduced density operator $\rho_R(t)$, we obtain the information entropies of the atomic operators σ_x, σ_y and σ_z in the form [33,34]

$$H(\sigma_\gamma) = -\sum_{i=1}^{4} P_i(\sigma_\gamma) \ln P_i(\sigma_\gamma), \gamma = x, y, z. \quad (8)$$

In Fig. (3), we investigate the effect of the qubit-photon coupling constants for the two qubits (g_1 and g_2) on the information entropies $H(\sigma_Z)$, $H(\sigma_Y)$, $H(\sigma_X)$

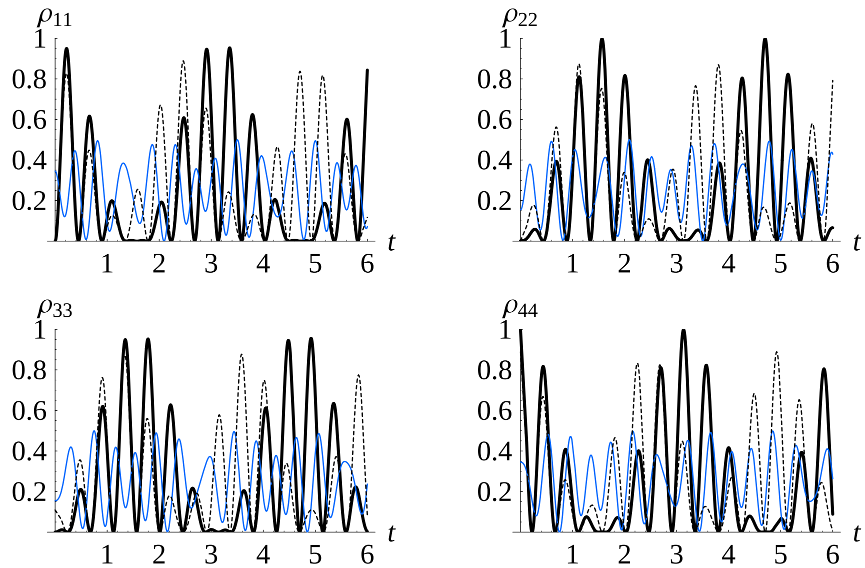

Figure 2: *These figures present the cases in which* $\omega = 1, g = 2, \grave{g} = 1.5$ $(g_1 = 7g_2), \Delta_1 = 0.5$, *where solid, blue and dot curves correspond respectively to,* $\Delta_2 = 0, \Delta_2 = 0.5$ *and* $\Delta_2 = 0.8$.

and the Von Neumann entropy $S(\rho_A)$. The information entropies $H(\sigma_Z)$, $H(\sigma_Y)$ and $H(\sigma_X)$ have regular and periodic oscillations. We show that when the qubit-photon coupling constants for the two qubits are equal ($g_1 = g_2$, $\grave{g} = 0$), the information entropy $H(\sigma_Z)$ starts from its minimum value, $H(\sigma_Z) = 0.85$, and increases until it reaches its maximum value at $H(\sigma_Z) = 1.25$. While $g_1 = 3g_2$, $\grave{g} = 1$, the information entropy $H(\sigma_Z)$ starts from its maximum value, $H(\sigma_Z) = 1.38$,and it has minimum value at $H(\sigma_Z) = 0.75$. When $g_1 = 7g_2$, $\grave{g} = 1.5$, the information entropy $H(\sigma_Z)$ starts from the value, $H(\sigma_Z) = 1.33$,and it has minimum value at $H(\sigma_Z) = 0.95$, and the number of periodic oscillations decreases. When the qubit-photon coupling constants for the two qubits are equal ($g_1 = g_2$, $\grave{g} = 0$), the information entropy $H(\sigma_X)$ starts from its minimum value, $H(\sigma_X) = 1.15$, and it has maximum value at $H(\sigma_X) = 1.35$. While $g_1 = 3g_2$, $\grave{g} = 1$, the information entropy $H(\sigma_X)$ has small oscillations. Also, when $g_1 = 7g_2$, $\grave{g} = 1.5$, the oscillations of the information entropy $H(\sigma_X)$ decreases remarkably. When the qubit-photon coupling constants for the two qubits are equal ($g_1 = g_2$, $\grave{g} = 0$), the information entropy $H(\sigma_Y)$ starts from its maximum value, $H(\sigma_Y) = 1.35$, and it decreases until it reaches its minimum value at $H(\sigma_Y) = 0.5$. While $g_1 = 3g_2$, $\grave{g} = 1$, the information entropy $H(\sigma_y)$ has small oscillations. Also, when $g_1 = 7g_2$, $\grave{g} = 1.5$, the oscillations of the information entropy $H(\sigma_Y)$ decrease. When the qubit-photon coupling constants for the two qubits are equal ($g_1 = g_2$, $\grave{g} = 0$), the Von Neumann entropy $S(\rho_A)$ starts from its minimum value, $S(\rho_A) = 0.25$, and it has maximum value at $S(\rho_A) = 0.45$. The oscillations of the Von Neumann entropy $S(\rho_A)$ has small phase. While $g_1 = 3g_2$, $\grave{g} = 1$, and ($g_1 = 7g_2$, $\grave{g} = 1.5$), the phase and the number of the oscillations of the Von Neumann entropy $S(\rho_A)$ increases remarkably. Also, the maximum and minimum values of the Von Neumann entropy $S(\rho_A)$ increase remarkably.

In Fig. (4), we investigate the effect of the detuning parameters (Δ_1 and Δ_2) on the information entropies $H(\sigma_Z)$, $H(\sigma_Y)$, $H(\sigma_X)$ and the Von Neumann entropy $S(\rho_A)$. The information entropies $H(\sigma_Z)$, $H(\sigma_Y)$ and $H(\sigma_X)$ have regular and periodic oscillations. We observe, when $\Delta_1 = 0.5, \Delta_2 = 0$, and ($\Delta_1 = 0.5, \Delta_2 = 0.8$), the information entropy $H(\sigma_Z)$ has many oscillations. While, when ($\Delta_1 = 0.5$, $\Delta_2 = 0.5$), the information entropy $H(\sigma_Z)$ has few number of oscillations. When $\Delta_1 = 0.5$, $\Delta_2 = 0$, and ($\Delta_1 = 0.5$, $\Delta_2 = 0.5$), the information entropy $H(\sigma_X)$ has many oscillations. While, when ($\Delta_1 = 0.5$, $\Delta_2 = 0.8$), the information entropy $H(\sigma_X)$ reaches fixed value at $H(\sigma_X) = 1.3$. When $\Delta_1 = 0.5$, $\Delta_2 = 0$, and ($\Delta_1 = 0.5$, $\Delta_2 = 0.8$), the information entropy $H(\sigma_Y)$ has the same maximum and minimum value and $H(\sigma_Y)$ starts from $H(\sigma_Y) = 1.2$. While, when $\Delta_1 = 0.5$, $\Delta_2 = 0.5$, the information entropy $H(\sigma_Y)$ starts from $H(\sigma_Y) = 1.4$ and has small phase of the oscillations. When $\Delta_1 = 0.5$, $\Delta_2 = 0$, and ($\Delta_1 = 0.5$, $\Delta_2 = 0.8$), the Von Neumann entropy $S(\rho_A)$ starts from its minimum value $S(\rho_A) = 0$ and has regular and periodic oscillations. While, when $\Delta_1 = 0.5$, $\Delta_2 = 0.5$, the Von Neumann entropy $S(\rho_A)$ starts from

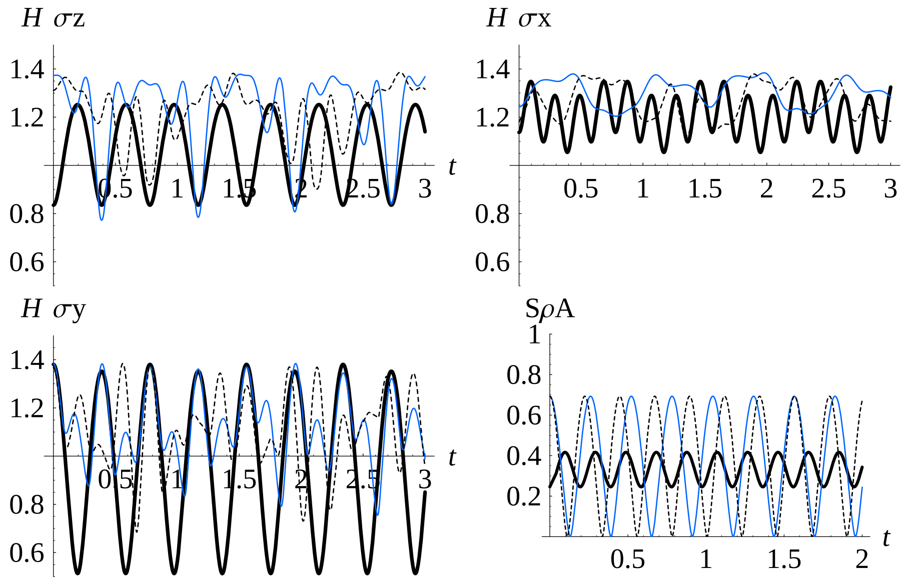

Figure 3: *These figures present the cases in which $\omega = 1, \Delta_1 = \Delta_2 = 0.5, g = 2$, where solid, blue and dot curves correspond, respectively, to $\grave{g} = 0 (g_1 = g_2), \grave{g} = 1\ (g_1 = 3g_2)$ and $\grave{g} = 1.5\ (g_1 = 7g_2)$*

its maximum value $S(\rho_A) = 0.7$ and and also has regular and periodic oscillations.

IV. Conclusion

In this paper, we analytically proposed an exact solution for two-qubit Rabi model in the general case. We studied the atomic occupation probabilities ρ_{11}, ρ_{22}, ρ_{33} and ρ_{44}, also, the information entropies $H(\sigma_Z)$, $H(\sigma_Y)$, $H(\sigma_X)$ and the Von Neumann entropy $S(\rho_A)$ for some special values of the qubit-photon coupling constants for the two qubits (g_1 and g_2) and the detuning parameters (Δ_1 and Δ_2). When the qubit-photon coupling constant g_2 increases the phase of the oscillations of the atomic occupation probabilities ρ_{11}, ρ_{44} and the information entropies $H(\sigma_Y)$ decreases, but the phase of the oscillations of the atomic occupation probabilities ρ_{22}, ρ_{33} and the Von Neumann entropy $S(\rho_A)$ increases. The occupation probabilities ρ_{11}, ρ_{22}, ρ_{33} and ρ_{44}, also, the information entropies $H(\sigma_Z)$, $H(\sigma_Y)$, $H(\sigma_X)$ and the Von Neumann entropy $S(\rho_A)$ have regular and periodic oscillations. When $\Delta_1 = 0.5$, $\Delta_2 = 0$, the oscillations of the occupation probabilities to a great extent look like the case of the spontaneous emission. The occupation probability ρ_{11} reaches the case of sudden death.

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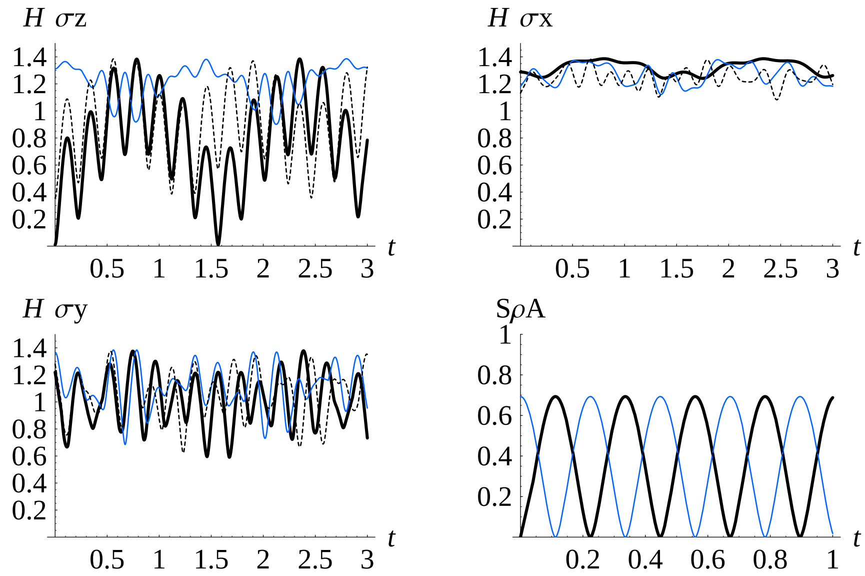

Figure 4: *These figures present the case in which g=2, $\grave{g} = 1.5$ ($g_1 = 7g_2$), $\Delta_1 = 0.5$, where solid, blue and dot curves correspond, respectively, to $\Delta_2 = 0$, $\Delta_2 = 0.5$ and $\Delta_2 = 0.8$.*

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