IIUM Journal of Economics and Management 12, no.2 (2004):

A FURTHER NOTE ON THE MATHEMATICAL DERIVATION OF TFP SOURCES

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1. INTRODUCTION

Abdullah (1997, 2000a and 2000b) has shown that the sources of Total Factor Productivity (TFP) can be decomposed into two major components, namely scale effect and technological change effect. However, it has been demonstrated that one of the components of TFP, i.e., the scale effect, can be further decomposed into three smaller components (Kuroda, 1989). Taking Kuroda's (1989) technique as the basis of extending our (1997, 2000a and 2000b) method of decomposing TFP, the present paper provides, with subtlety, two more credentials to the study of TFP. First, by incorporating the latest method of decomposing the scale effect into the TFP decomposition analysis, it suggests that we can go deeper into ascertaining the sources of TFP growth of a production unit. Second, accordingly we can figure out with a high degree of certainty the source that contributes most to the TFP growth of a production unit. In this paper the component is associated with the scale effect of the TFP component.

To avoid any confusion that may arise if we by-passed many important equations that are closely related to the derivation of TFP and its components, in this paper we choose to reproduce most of the procedures that were discussed in the previously mentioned published papers. We reiterate here that the translog cost function is still the function from which the sources of TFP are derived.

The plan of the paper is as follows. In section 2, we will systematically show how the decomposition analysis of the sources of

TFP growth, i.e., scale economies and technological change, are disentangled. In section 3, the decomposition based on the translog cost function will be discussed. Next, in section 4, we will present the procedures that are used to derive the scale effect component of TFP sources. Specifically, in this paper the three major sources of TFP growth that are subset to the scale effect will be integrated into the analysis. They are: (i) output changes of scale economies; (ii) factor price changes of scale economies; and (iii) technological change of scale economies. We note, in passing, that readers who have abyssal interests concerning the detailed explanations for each component of the scale economies (SCE) can refer to Greene (1983) and Stevenson (1980). In section 5, the summary and concluding remarks are given.

2. DECOMPOSITION ANALYSIS

We assume the production unit that is under scrutiny is characterized by a production function satisfying the usual regularity conditions,[1]

(1) $$Y = f(X, T)$$

where X is a vector of m inputs, T is time, which indicates the effect of technological change, and Y denotes output. Assuming that input prices, W_j where $j = (1, 2, 3, \ldots m)$, are exogenously determined, the dual cost function may be written as

(2) $$C = C(W_1, W_2, \ldots, W_m, Y, T)$$

where production cost (C) is a function of the input prices, the level of output (Y), and time (T). We further assume that factor markets are competitive and that the production unit is willing to supply all output demanded at any given price. Thus, input prices and output are treated as exogenous variables while input levels are endogenous.

Now, logarithmically differentiating (2) with respect to (w.r.t.) time (*T*), we can decompose the rate of growth of total cost into its source components:

$$(3) \qquad \frac{\dot{C}}{C} = \left[\sum_{j=1}^{m} \frac{\partial \ln C}{\partial \ln W_j} \frac{\dot{W}_j}{W_j} \right] + \left[\frac{\partial \ln C}{\partial \ln Y} \frac{\dot{Y}}{Y} \right] + \left[\frac{\partial \ln C}{\partial T} \right]$$

The variables with dots on top denote a differentiation w.r.t. time (*T*). In words, the rate of growth of total cost ($\frac{\dot{C}}{C}$) can be expressed as the cost elasticity weighted average of rates of growth of input prices, plus the scale weighted rate of growth of output, plus the rate of cost diminution due to technological change.

Applying Shephard's lemma to the logarithmic partial derivative appearing in (3), we then obtain the following relations:

$$(4) \qquad \sum_{j=1}^{m} \frac{\partial \ln C}{\partial \ln W_j} = \sum_{j=1}^{m} \frac{X_j W_j}{C} = \sum_{j=1}^{m} S_j = 1$$

where $S_j = X_j W_j / C$ denotes the cost share of the j^{th} input.

Next, we define the elasticity of cost w.r.t. output (*Y*), e_{CY} as

$$(5) \qquad \frac{\partial \ln C}{\partial \ln Y} = \frac{\partial C}{\partial Y} \frac{Y}{C} = \varepsilon_{CY}$$

Equation (5) is used as an indicator to measure the returns to scale. The ε_{CY} indicates increasing returns to scale, constant returns to scale, or decreasing returns to scale as $\varepsilon_{CY} < 1$, $\varepsilon_{CY} = 1$, or $\varepsilon_{CY} > 1$, respectively.

We define λ as the rate of growth of cost diminution,

(6)

Hence, collecting (4), (5) and (6) and substituting them into (3) yields

$$\frac{\dot{C}}{C} = \left[\sum_{j=1}^{m} S_j \frac{\dot{W}_j}{W_j}\right] + \left[\varepsilon_{CY} \frac{\dot{Y}}{Y}\right] + [\lambda] \tag{7}$$

Finally, given (4), and then differentiating the factor input costs, $C = \sum_{j=1}^{m} W_j X_j$ w.r.t. time (*T*), dividing by *C* and rearranging, we get

$$\sum_{j=1}^{m} S_j \frac{\dot{W}_j}{W_j} = \frac{\dot{C}}{W} - \sum_{j=1}^{m} S_j \frac{\dot{W}_j}{W_j} \tag{8}$$

2.1 TOTAL FACTOR PRODUCTIVITY (TFP)

Before proceeding further, we introduce the mathematical approach for computing the TFP growth rate and subsequently relate it to (8). To begin with, we denote index of output (*Y*) where the rate of growth of which is expressed as

$$\frac{\dot{Y}}{Y} = \sum_{i=1}^{n} N_i \frac{\dot{Y}_i}{Y_i} \tag{9}$$

where $N_i = P_i Y_i / \sum_{i=1}^{n} P_i Y_i = P_i Y_i / R$, and P_i and Y_i respectively price and quantity of output *i*; $R = \sum_{i=1}^{n} P_i Y_i$, the total revenue; and $\dot{Y}/Y$, the rates of growth of output *i*.

An analogous index of the quantity of total input, say *X*, is expressed as

$$(10) \qquad \frac{\dot{X}}{X} = \sum_{j=1}^{m} S_j \frac{\dot{X}_j}{X_j}$$

where $S_j = W_j X_j / \sum_{j=1}^{m} W_j X_j = W_j X / C$, and W_j and X_j are, respectively, the price and quantity of input *j*; $C = \sum_{j=1}^{m} W_j X_j$, the total cost; and $\dot{X} / X$ the rates of growth of input *j*. These two quantity indexes may be regarded as a family of Divisia quantity indexes.

We continue to define TFP and *P* in terms of Divisia index numbers. *P* is defined as the ratio of total output to the quantity of total input:

$$(11) \qquad P = \frac{Y}{X}$$

The rate of growth of TFP [*P/P*] is defined as

$$(12) \qquad \frac{\dot{P}}{P} = \frac{\dot{Y}}{Y} - \frac{\dot{X}}{X}$$

2.2 DECOMPOSITION ANALYSIS AND TFP - A TIE-UP

Having equations (1) to (12) at our disposal, we can now establish a link between the decomposition analysis and TFP. This can be done by, first, substituting (7) into (8). After rearranging we obtain

$$(13) \qquad \varepsilon_{CY} \frac{\dot{Y}}{Y} + \lambda - \sum_{j}^{4} S_j \frac{\dot{X}_j}{X_j} = 0$$

Second, substituting (10) into (13), we obtain

$$\varepsilon_{CY}\frac{\dot{Y}}{Y}+\lambda-\frac{\dot{X}}{X}=0 \tag{14}$$

Finally, using definition $\frac{\dot{P}}{P}=\frac{\dot{Y}}{Y}-\frac{\dot{X}}{X}$ and rearranging, (14) becomes

$$\frac{\dot{P}}{P}=(1-\varepsilon_{CY})\frac{\dot{Y}}{Y}-\lambda \tag{15}$$

If constant returns to scale exist, then $(1-\varepsilon_{CY}) = 0$, implying $\dot{P}/P=-\lambda$. This expression, as has been noted in section 1, is tantamount to the conventional growth accounting measurement of TFP which is equivalent to the negative rate of cost diminution. If, however, we conjecture that scale effects are present ($\varepsilon_{cY}\neq 1$), then it turns out that the conventional measure estimates of the growth rate of TFP should include both scale effects and technological change effects. This is to say that (15) holds.

3. TRANSCENDENTAL LOGARITHMIC (TRANSLOG) COST FUNCTION: THE PROPOSED FUNCTION TO BE EMPLOYED IN EMPIRICAL WORK

In order to compute the terms in the decomposition equation (15), we specify the cost function in transcendental logarithmic, or for brevity translog, form. Assuming that the translog cost function is represented by

$$\ln C=\alpha_0+\sum_{j=1}^{4}\alpha_Y\ln W_j+\alpha_Y\ln Y+\alpha_T T \tag{16}$$

$$+\frac{1}{2}\sum_{j=1}^{4}\sum_{k=1}^{4}\gamma_{jk}\ln W_j\ln W_k+\frac{1}{2}\gamma_{YY}(\ln Y)^2$$

$$+\sum_{j=1}^{4}\gamma_{jT}\ln W_j T+\sum_{j=1}^{4}\delta_{Yj}\ln W_j \ln Y$$

$$+\delta_{YT}\ln Y(T)+\frac{1}{2}\gamma_{TT}(T)^2$$

where all subscripts remained as they were before.

The cost-share S_j is derived through Shephard's lemma as

(17) $$S_j=\alpha_j+\sum_{k=1}^{4}\gamma_{jk}\ln W_k+\delta_{jY}\ln Y+\gamma_{jT}T$$

Next, cost elasticity is defined as $\varepsilon_{CY}=\partial \ln C/\partial \ln Y$ and if applied to equation (16) will give

(18) $$\varepsilon_{CY}=\partial \ln C/\partial \ln Y=\alpha_Y+\gamma_{YY}\ln Y+\delta_{Yj}\ln W_j+\delta_{YT}T$$

Equation (18) provides information on returns to scale. If $\alpha_Y=1$ $\gamma_{YY}=0$ and $\delta_{Yj}=0$ (j = L, M, U, B), then ε_{CY} = 1, signifying constant returns to scale. If, however, $\alpha_Y>1$ or $\alpha_Y<1$ then ε_{CY} > 1 or ε_{CY} < 1, signifying decreasing returns to scale or increasing returns to scale, respectively. Using (18), we will analyze each component impact on the Scale Economies (SCE). They are: (i) output changes of scale economies; (ii) factor prices changes of scale economies; (iii) technological change of scale economies.

4. DERIVATION OF THE SUBSETS SCALE EFFECT COMPONENT OF TFP SOURCES

Using (18) we can compute the output and factor input prices components of scale economies.

a. Output Changes Component of Scale Economies

The output component is computed as

$$(19)\qquad \frac{\partial SCE}{\partial \ln Y} = \frac{\partial(1-\varepsilon_{CY})}{\partial \ln Y} = -\frac{\partial \varepsilon_{CY}}{\partial \ln Y} = \gamma_{YY}$$

b. Factor Input Prices Changes Component of Scale Economies

The factor input prices component is computed as

$$(20)\qquad \frac{\partial SCE}{\partial \ln W_j} = \frac{\partial(1-\varepsilon_{CY})}{\partial \ln W_j} = -\frac{\partial \varepsilon_{CY}}{\partial \ln W_j} = \delta_{jY}$$

In both cases, any positive (negative) values indicate that increases in the corresponding variable lead to higher (lower) scale economies.

(c) Technological Change Component of Scale Economies

The technological change component of SCE is derived by differentiating (18) w.r.t. time (*T*) to yield

(21)[2]

$$\frac{\partial SCE}{\partial T} = \frac{\partial(1-\varepsilon_{cY})}{\partial T} = -\frac{\partial \varepsilon_{cY}}{\partial T}$$

$$= -\partial\left(\alpha_Y + \gamma_{YY}\ln Y + \delta_{Yj}\ln W_j + \delta YT\right)/\partial T$$

$$= -\delta_{YT}$$

In order to derive efficient scale, we add the following information, that is, the efficient scale at which average cost reaches its minimum and returns to scale equals 1 ($\varepsilon_{CY} = 1$), to equation (21) and rearranging will give

$$(22)\qquad \ln Y = 1 - \alpha_Y - \delta_{Yj}\ln W_j + \delta_{YT}T/\gamma_{YY}$$

Next, differentiating (22) w.r.t. time (*T*) yields

$$\frac{\partial \ln Y}{\partial T} = \frac{\delta_{Yj}}{\gamma_{YY}} \quad (23)$$

The efficient scale (ES), as shown in (23) grows by a proportional rate $\delta_{Yj}^{rl} / \gamma_{YY}^{rl}$. In this case ES of a production unit is computable from the parameter estimates of equation (16). Converse to the interpretations of $\partial SCE / \partial \ln Y$ and $\partial SCE / \partial \ln W_j$, the interpretation of $\partial SCE / \partial T$ is as follows. A negative (positive) value of $\partial SCE / \partial T$ indicates that an increase in technological change leads to higher (lower) degree of scale economies.

If a comparison is made between the parameter estimates obtained for $\partial SCE / \partial T$ and Efficient Scale (ES), any negative values of the derivative $\partial SCE / \partial T$ below the ES and positive values above the ES imply reduction in the slope of the cost curve (Greene, 1983).[3]

5. SUMMARY AND CONCLUDING REMARKS

Our main objective in writing this paper was to investigate the sources of a firm's TFP growth. For this reason we established a method which made possible for us to firstly, link the TFP analysis with the theory of production and secondly, to disentangle the sources of TFP growth into scale and technological change effects. In pursuing this objective a translog cost function, the latest methodological framework to measure production technology, was employed. A contribution of the present paper is that it extended the previous model of decomposing the TFP growth where the scale effect component has been further decomposed into three sub-components. An example of how the present technique of decomposing the TFP used in empirical works is available in Abdullah (2003).

ENDNOTES

1. For a detailed discussion on regularity conditions see for example Binswanger (1974) or Diewert (1978).

2. Alternatively, if the symbol ∂ is cancelled out from both, the numerator and denominator, equation (22) can be written as

We note here that this is the way Kuroda (1989) expressed the technological change component of SCE.

3. As been pointed out by Greene (1983), Stevenson's (1980) interpretation of the expression $TS_C = \partial SCE / \partial T$, which the latter author uses to measure the technological scale bias, was misleading. Changes in ε_{CY} according to Greene, is related more closely to changes in slope of the average cost curve than to changes in its location.

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