

SINGULAR VECTOR FOR A SHALLOW WATER MODEL

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ABSTRACT: The atmosphere is a chaotic dynamical system, small errors in the initial condition of an atmospheric model can lead to growing errors in the forecast. In order to reduce uncertainty in the forecast which occur from using only one initial condition from many possible initial conditions, more initial conditions are required. In practice, however, there is only single observation available for the initial condition. In this paper the singular vector has been applied to a single level shallow water model of the atmosphere to generate more initial conditions from one observation.

ABSTRAK: Atmosfera ialah satu sistem dinamik yang huru-hara. Kesilapan kecil dalam kondisi awal model atmosfera boleh membawa kepada kesilapan yang semakin meningkat di dalam ramalan. Dalam usaha untuk mengurangkan ketidakpastian dalam ramalan yang berlaku dari menggunakan hanya satu kondisi awal, lebih banyak kondisi awal diperlukan. Secara praktikalnya, walau bagaimanapun, hanya terdapat satu pemerhatian yang sedia ada untuk kondisi awal. Kajian ini menerangkan vektor tunggal telah digunakan untuk model air cetek tahap atmosfera untuk menghasilkan lebih banyak kondisi awal dari satu pemerhatian.

KEYWORDS: *shallow water model; tangent linear model; adjoint linear model; singular vector*

1. INTRODUCTION

The results from numerical models of the atmosphere can sometimes contain large errors. A reason for this lies in the nature of the atmosphere in which very small errors in the initial conditions can lead to large errors in the forecast. Implying a perfect forecast system cannot be created because every detail of the initial state of the atmosphere cannot be observed. To cope with this uncertainty, ensemble forecasting which is a collection of difference forecasts and all valid at the same forecast time is applied. Kalney [1] discussed a simple model by using the singular vector technique to create ensemble forecasting. Wei and Frederiksen [2] used Lyapunov vectors and singular vectors to examine errors growth and predictability in atmospheric models. Khade and Hansen [3] found that the singular error growth in the most parts of the attractor is rapid, in that errors can grow almost by a factor of 3 over one step.

In this paper the singular vector method is applied to a linearized shallow water model of the atmosphere to generate more initial conditions from the initial data.

2. METHODOLOGY

2.1 The Shallow Water Model

The shallow water model is a simplified version of the equations of motion. The momentum, hydrostatic and continuity equations are given below [4].

$$\frac{du}{dt} - fv + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0 \quad (1)$$

$$\frac{dv}{dt} + fu + \frac{1}{\rho} \frac{\partial p}{\partial y} = 0 \quad (2)$$

$$\frac{\partial p}{\partial z} + \rho g = 0 \quad (3)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (4)$$

where ρ is the density of the atmosphere, p is the pressure, f is the Coriolis parameter which is given by $f = 2\Omega \sin \theta$ (Ω is the angular velocity of the earth, θ is the latitude), g is the acceleration due to gravity and u, v and w are the speed of fluid in the x, y and z directions, respectively.

The shallow water model consists of a set of three equations with three dependent variables (u, v, ϕ).

$$\frac{du}{dt} - fv + g \frac{\partial \phi}{\partial x} = 0 \quad (5)$$

$$\frac{dv}{dt} + fu + g \frac{\partial \phi}{\partial y} = 0 \quad (6)$$

$$\frac{d\phi}{dt} + \phi \nabla_H \cdot V = 0 \quad (7)$$

where ϕ is the geopotential height.

The vertical velocity does not appear explicitly. The total time derivative is given by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}. \quad (8)$$

2.2 The Singular Vector Method

Singular vector (SV) method is explained here according to Kalney [1]. The initial conditions for the ensemble prediction system (EPS) are created to represent the uncertainties in the operational analysis. They are made by adding to the operation analysis perturbations which produce the fastest growth during the first stage of the forecast period, defined using the singular vector technique. The purpose is to find those perturbations to a given initial state which grow most rapidly [5]. The calculation of the singular vectors is the tangent - linear version of the full (non -linear) operational forecast

model. A second model, the adjoint of the tangent linear model is also needed in the singular vector calculation.

a) The Tangent Linear Model Method (TLM)

Consider a non linear model

$$x(t) = M[x(t_0)] \quad (9)$$

where $x(t_0)$ is the state at the initial time and $x(t)$ is the state at time t and M is the time integration of the numerical scheme from the initial condition to time t .

The tangent linear model (TLM) is defined such that

$$L = \frac{\partial M}{\partial x} \quad (10)$$

where L is the tangent linear model of M .

So the TLM evolves a perturbation in time;

$$y(t) = L(t_0, t)y(t_0) \quad (11)$$

where $y(t_0)$ is the perturbation state at the initial time $t = 0$ and $y(t)$ is the perturbation state at time t [3].

b) The Adjoint of the Tangent Linear Model

The adjoint tangent linear model is the transpose of the tangent linear model.

c) The Singular Vector

Singular value decomposition theory indicates that for any matrix L there exist two orthogonal matrices U, V such that

$$U^T L V = S \quad (12)$$

where $S = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \sigma_n \end{bmatrix}$ and $U U^T = I, V V^T = I$

S is a diagonal matrix whose elements are the singular values of L and I is the identity matrix.

So that

$$L v_i = \sigma_i u_i. \quad (13)$$

The vector (v_1, \dots, v_n) are the right singular vectors of L .

Consequently

$$L^T u_i = \sigma_i v_i \quad (14)$$

The vector (u_1, \dots, u_n) are the left singular vectors of L .

Therefore, the initial singular vectors can be obtained as the eigenvectors of $L^T L$, the final singular vectors can be obtained as the eigenvectors of LL^T and eigenvalues are the squares of the singular values.

Consider a basic trajectory

$$y(t_0) = \sum_{i=1}^n \langle y(t_0), v_i \rangle v_i \quad (15)$$

$$y(t_1) = \sum_{i=1}^n \langle y(t_1), u_i \rangle u_i. \quad (16)$$

Applying the tangent linear model L in (11) and (15) and rearrange in the form

$$\langle y(t_1), \sigma_i u_i \rangle = \langle y(t_0), v_i \rangle \quad (17)$$

Applying the adjoint of the tangent linear model L^T in (11) and (16)

$$\langle y(t_0), \sigma_i v_i \rangle = \langle y(t_1), u_i \rangle \quad (18)$$

Applying $L^T L$ by running tangent linear model forward in time, and then the adjoint backward in time.

$$\langle y(t_1), \sigma_i u_i \rangle = \sigma_i^2 \langle y(t_0), v_i \rangle \quad (19)$$

2.3 The Numerical Method of the Shallow Water Model

The forward in time center in space finite difference method is used to solve the shallow water equations on Arakawa - A grid.

a) Discretized the Shallow Water Model

$$\begin{aligned} u_i^{j+1} &= u_i^j - \alpha u_i^j (u_{i+1}^j - u_{i-1}^j) - \beta v_i^j (u_{i+1}^j - u_{i-1}^j) + \Delta t f v_i^j - \alpha g (\phi_{i+1}^j - \phi_{i-1}^j) \\ v_i^{j+1} &= v_i^j - \alpha u_i^j (v_{i+1}^j - v_{i-1}^j) - \beta v_i^j (v_{i+1}^j - v_{i-1}^j) - \Delta t f u_i^j - \beta g (\phi_{i+1}^j - \phi_{i-1}^j) \\ \phi_i^{j+1} &= \phi_i^j - \alpha u_i^j (\phi_{i+1}^j - \phi_{i-1}^j) - \beta v_i^j (\phi_{i+1}^j - \phi_{i-1}^j) - \alpha \phi_i^j (u_{i+1}^j - u_{i-1}^j) - \beta \phi_i^j (v_{i+1}^j - v_{i-1}^j) \end{aligned} \quad (20)$$

where $\alpha = \frac{\Delta t}{2\Delta x}$, $\beta = \frac{\Delta t}{2\Delta y}$

b) Initial Condition

For this research the initial conditions are

$$u_0 = -g \frac{d\phi}{dy} / f \quad (21)$$

$$v_0 = g \frac{d\phi}{dx} / f \quad (22)$$

$$\phi_0 = \phi_{Wave} + \phi_{Bar} + \phi_{Ref} \quad (23)$$

where

$$\phi_{Wave} = \phi_{Amp} * \sin(2 * \pi * m_x * X / L_x) * \cos(m_y * \pi * (Y - y_0) / L_y)$$

$$\phi_{Bar} = -\left(\frac{u_{Bar}}{g}\right) * (f_0 * (Y - y_0) + 0.5 * \beta * (Y - y_0)^2) \quad \phi_{Ref} = 5500 \text{ is the constant reference}$$

geopotential height, $\phi_{Amp} = 500$ is the constant amplitude geopotential height. $m_x = 1$ and $m_y = 1$ are wave numbers in x and y directions, respectively. L_x and L_y are channel lengths in x and y directions, respectively.

c) Boundary Condition

The boundary condition in the east-west direction is a cyclic boundary and the north-south direction is an updated boundary.

d) Experiment

The domain of the model is 32×10^6 meters in x-direction and 8×10^6 meters in y-direction, grid size in x-direction is 8×10^5 meters and grid size in y-direction is 8×10^5 meters, time step is 600 seconds. The output interval from the forecast model is 6 hours.

2.4 Step of Calculation the Singular Vector

Step1: The tangent linear model

From (10) for the shallow water model is calculated as

$$L = \frac{\partial M}{\partial U} = \begin{bmatrix} \frac{\partial^2 u}{\partial x \partial t} + u \frac{\partial^2 u}{\partial x \partial x} + \frac{\partial u}{\partial x} + v \frac{\partial^2 u}{\partial x \partial y} & \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} - f & g \frac{\partial^2 z}{\partial x \partial x} \\ \frac{\partial v}{\partial x} + f & \frac{\partial^2 v}{\partial t \partial t} + u \frac{\partial^2 v}{\partial x \partial x} + v \frac{\partial^2 v}{\partial y \partial y} + \frac{\partial v}{\partial y} & g \frac{\partial^2 z}{\partial x \partial y} \\ \frac{\partial z}{\partial x} + z \frac{\partial^2 u}{\partial x \partial x} & \frac{\partial z}{\partial y} + z \frac{\partial^2 v}{\partial y \partial y} & \frac{\partial^2 z}{\partial x \partial t} + u \frac{\partial^2 z}{\partial x \partial x} + v \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} \end{bmatrix} \quad (24)$$

The discretized form of (24) is then used to calculate the tangent linear model at each of grid point.

Step2: The adjoint linear tangent model is the transpose of the linear tangent model.

Step3: Calculating the singular vector and singular value from (13) and (14) have to be repeated many times to generate several perturbations.

The maximum norm of perturbations and minimum norm of perturbations are then added to the initial condition to generate the new initial conditions.

In this paper the norm of vector is defined as

$$\|y_i\| = \sqrt{|u|^2 + |v|^2 + |\phi|^2} \quad (25)$$

For example, at a point where the initial perturbations $y(t_0) = \begin{bmatrix} -0.1871 \\ 0.4344 \\ 277.2987 \end{bmatrix}$ a single cycle of L and

L^T transformations result in $y(t_1) = \begin{bmatrix} 1.7712 \\ 0.0629 \\ 3.8762e^{-10} \end{bmatrix}$.

3. RESULTS AND DISCUSSION

3.1 Result of the shallow water model

Figure 1 shows the velocity and geopotential field following (21) and (22). Figure 2 shows the result of forecast at 6 hours.

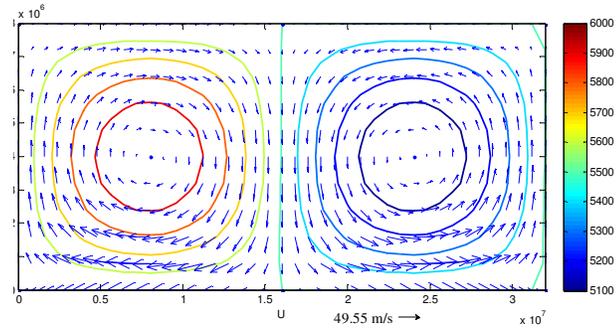


Fig. 1: The initial state of the velocity field (vectors) and geopotential heights (contour lines).

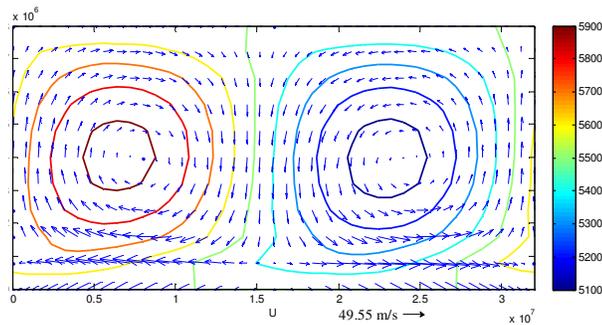


Fig. 2: The forecast state of the velocity field (vectors) and geopotential heights (contour lines) at 6 hours.

Table 1:
 Examples of singular vectors and singular values of perturbation at grid point (2,3).

State	Singular vector			Perturbation
	v_1	v_2	v_3	
I	$\begin{bmatrix} 0.859 \\ -0.511 \\ -6.907e^{-5} \end{bmatrix}$	$\begin{bmatrix} 0.512 \\ 0.859 \\ 4.979e^{-5} \end{bmatrix}$	$\begin{bmatrix} 3.386e^{-5} \\ -7.813e^{-5} \\ 0.999 \end{bmatrix}$	$\begin{bmatrix} -0.187 \\ 0.434 \\ 277.298 \end{bmatrix}$
N	1	1	1	277.2992
F	$\begin{bmatrix} 0.221 \\ -0.131 \\ -1.778e^{-5} \end{bmatrix}$	$\begin{bmatrix} 1.598 \\ 2.683 \\ 0.0001 \end{bmatrix}$	$\begin{bmatrix} 2.422e+7 \\ -5.589e+7 \\ 7.154e+7 \end{bmatrix}$	$\begin{bmatrix} 1.771 \\ 0.063 \\ 3.877e^{-10} \end{bmatrix}$
N	0.2574	3.12273	7.154e+11	1.77234

In Table 1, I denotes the initial state of singular vector, F is the final state of singular vector and N is the norm of singular vector. The singular vector at the grid point in Table 1 can be represented as vectors as shown in Fig. 3–5.

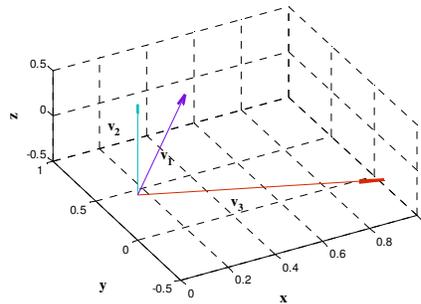


Fig. 3: v_i are the initial time singular vectors; x , y and z axes represent u , v , and ϕ , respectively.

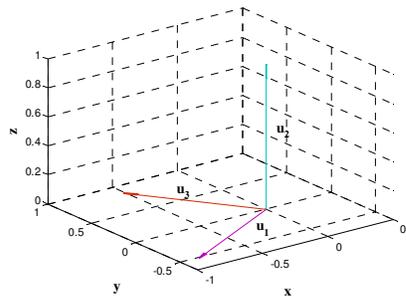


Fig. 4: U_i are the final time singular vectors with $\sigma_1 = 0.2574$, $\sigma_2 = 3.1227$ and $\sigma_3 = 7.1537e11$; x , y and z axes represent u , v , and ϕ , respectively.

The tangent linear model stretches (or shrinks) and rotates each of the perturbation vectors. At each step the initial state can be transformed to the final state as shown in Fig. 3. Figure 4 shows the adjoint tangent linear model stretches (or shrinks) and rotates for each of the perturbation vectors.

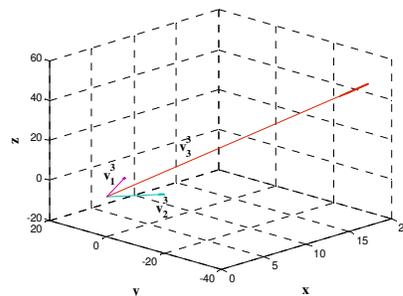


Fig. 5: v_i are the final time singular vectors with $\sigma_1^2 = 0.066$, $\sigma_2^2 = 9.7512$ and $\sigma_3^2 = 5.1175e+23$; x , y and z axes represent u , v , and ϕ , respectively.

4. CONCLUSION

In this paper a singular vector is applied at a grid point in linearized shallow water wave model. A single vector transformation results in a significant the change in the norm of the initial perturbation. The method can be used to generate initial states to an ensemble shallow water model for ensemble forecasts.

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