ON THE SETTLING AND RESPONSE TIMES OF UNDERDAMPED SECOND-ORDER SYSTEMS

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ABSTRACT: The normalized settling time (ts /τ) values of oscillatory 2nd-order systems, when subjected to a step-change forcing function (SCFF), depend on the sensitivity of the measuring instrument employed to indicate the response (± x%). An attempt is made to mathematically relate ts /τ to ± x% utilizing the exact, and a simplified, expression for the lower boundary of the decay envelope (LBDE). The two obtained relationships were tested against the actual ts /τ values for a settling band range of ±1% ≤ ±x% ≤ ±6%, covering a damping coefficient range of 0.1 ≤ ζ ≤ 0.65. Although the relationships are not exact, their general trend is a marginal overestimation of ts /τ. The relationship based on the simplified LBDE was chosen for being simpler and slightly more accurate of the two. This led to a suggested distinction between ts /τ and the normalized response time (tR /τ) with the latter assigned the value 5/ ζ . The ratio ts / tR can thus be readily established for any ± x% value.

ABSTRAK: Masa enapan ternormal (t_s/τ) nilai ayunan system terbit kedua, apabila fungsi memaksa ubah berperingkat (step-change forcing function (SCFF)) dijalankan ke atasnya, bergantung kepada kepekaan alat pengukur yang digunakan untuk mengukur respons (± x%). Satu percubaan dijalankan secara matematik untuk mengaitkan t_s/τ to ± x% dengan mempergunakan ekspresi yang tepat dan mudah, pada sempadan bawah sampul reputan (lower boundary of the decay envelope (LBDE)). Dua hubungan yang diperolehi dikaji terhadap nilai t_s/τ sebenar untuk julat jalur enapan ±1% ≤ ±x% ≤ ±6%, melingkungi julat pekali redaman 0.1 ≤ ζ ≤ 0.65. Walaupun hubungannya tidak tepat, trend umum merupakan penganggaran marginal t_s/τ. Hubungan berdasarkan LBDE adalah berdasarkan LBDE yang telah dipermudahkan, ia dipilih kerana ianya senang dan agak tepat antara keduanya. Ini mendorong kepada perbezaan yang disarankan antara t_s/τ dan waktu respons ternormal (t_R/τ), dengan nilai 5/ ζ yang ditetapkan kemudiannya.

KEYWORDS: *process control; instrumentation; mathematical modelling; transient response; 2nd-order system*

1. INTRODUCTION

An underdamped second order system with $0 < \zeta \leq 0.65$, when subjected to a SCFF undergoes a response which is significantly oscillatory. Under such condition, the settling time (also called the response or recovery time) is defined as the time required for the normalized response to enter the ±5% band of the step change magnitude. Other definitions for t_s exist; notably, that related to the ±2% band or as determined by the sensitivity of the measuring instrument [1-5].

2. ESTIMATION OF SETTLING TIME

Pollard [2] pointed out that owing to the arbitrary nature of the settling band limits (due to the specific sensitivity of the measuring instrument employed), a mathematical definition for t_s is not possible. He concluded that it can be easily measured from the response curve of a recording instrument (i.e. a posteriori). In spite of the aforementioned viewpoint, an estimate of t_s value a priori is an advantage in many instances, e.g. in the design and analysis of control loops. The normalized response of an underdamped 2nd-order system to a SCFF of magnitude A is,

$$\frac{Y(t)}{AK} = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\frac{\zeta}{\tau}t} \sin\left(\frac{\sqrt{1-\zeta^2}}{\tau} t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \quad (1)$$

For a ± x% settling band, its limits would correspond to Y(t)/AK values of (1 + 0.01 x) and (1 - 0.01 x) respectively. This renders the value of the second term of Eq. (1) equal to 0.01x in absolute value. Therefore, t_s/τ is the shortest normalized time which satisfies this condition; provided that the second term will not exceed |0.01 x| for $t/\tau > t_s/\tau$.

The exact expression for the lower boundary of the decay envelope (LBDE) related to the normalized response represented by Eq.(1) is $(1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\frac{\zeta}{\tau}t})$ (Ogata [5]). Pollard [2], however, gave it as $(1 - e^{-\frac{\zeta}{\tau}t})$; neglecting ζ^2which is justifiable for small values of ζ. Pollard further pointed out that $(1 - e^{-\frac{\zeta}{\tau}t})$ is the normalized response of a 1st–order system, whose time constant is τ/ζ, to a SCFF.

These two expressions for the LBDE were utilized to obtain the following equations for the estimation of t_s/τ as related to a –x% settling band limit. Hence,

$$\frac{t_s}{\tau} = \frac{-\ln(0.01\,x)}{\zeta} \quad (2)$$

Corresponding to $(1 - e^{-\frac{\zeta}{\tau}t})$ LBDE , and

$$\frac{t_s}{\tau} = \frac{-\ln(0.01\,x)}{\zeta} - \frac{1}{2\zeta}\ln\left(1-\zeta^2\right) \quad (3)$$

Corresponding to $(1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\frac{\zeta}{\tau}t})$ LBDE.

Eq.'s (2) and (3) were tested against the actual t_s/τ values for settling bands ranging from ±1% to ±6% over the range $0.1 \leq \zeta \leq 0.65$. The results are shown in Fig.'s (1) to (6). The two equations generally overestimate t_s/τ but give reasonably close values to the actual ones. Their respective values of t_s/τ were too close to be distinguishable on the same graph; which necessitated the use of separate plots for each ±x% value.

The percentage error as defined by;

$$\%\ \text{error} = \frac{(t_s/\tau)_{calc.} - (t_s/\tau)_{act.}}{(t_s/\tau)_{act.}} \times 100 \quad (4)$$

ranged from -8.35% to 26.86% with an average of 6.93% for Eq.(2) and from 0.00% to 31.62% with an average of 10.00% for Eq.(3) over the whole range tested. Therefore, it may be concluded that Eq.(2) is the better one for being simpler and marginally more accurate.

For settling bands of ±2% and ±5%, Eq.(2) gives $t_s/\tau = 3.912/\zeta$ and $t_s/\tau = 2.996/\zeta$ respectively. These two results correspond to the familiar expressions of $t_s/\tau = 4/\zeta$ for ±2% settling band and $t_s/\tau = 3/\zeta$ for ±5% settling band mentioned in textbooks as approximate relationships (e.g. Ogata [5]); further validating the approach presented here.

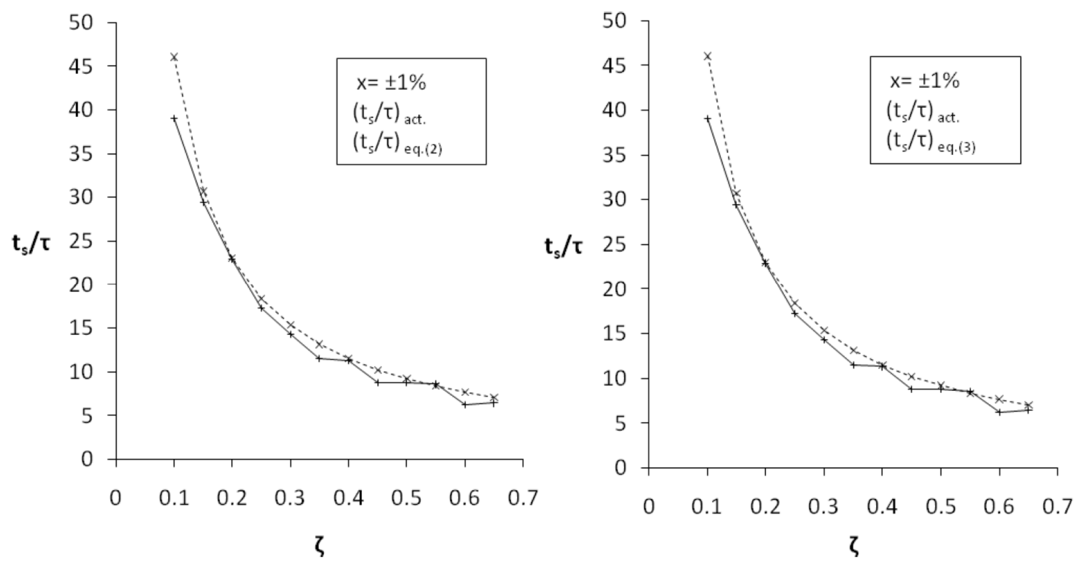

Fig. 1: Actual and calculated (t_s/τ) vs. ζ for ±1% settling band.

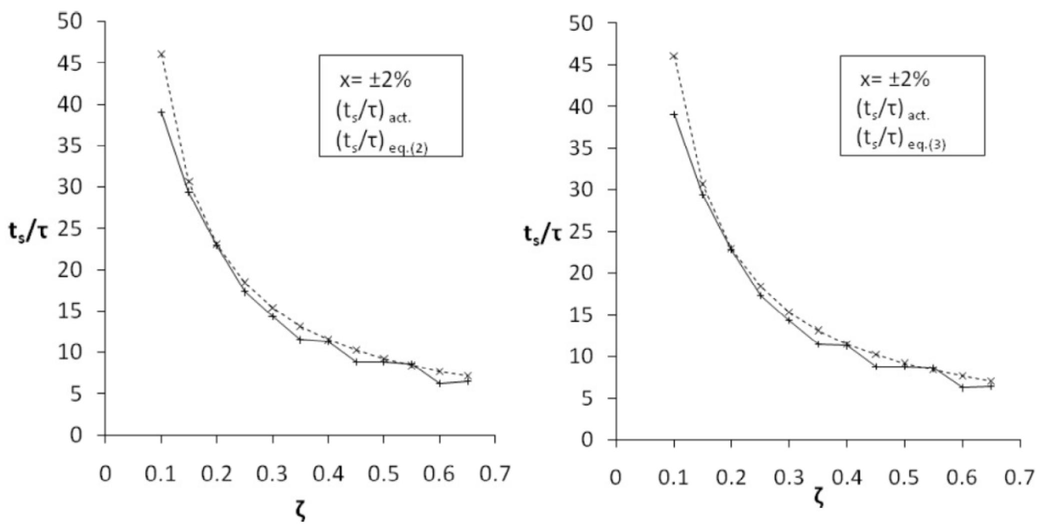

Fig. 2: Actual and calculated (t_s/τ) vs. ζ for ±2% settling band.

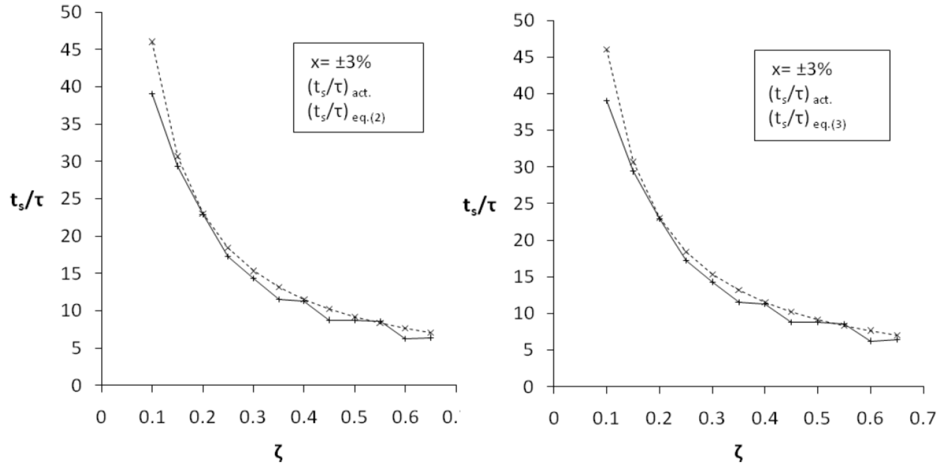

Fig. 3: Actual and calculated (t_s/τ) vs. ζ for ±3% settling band.

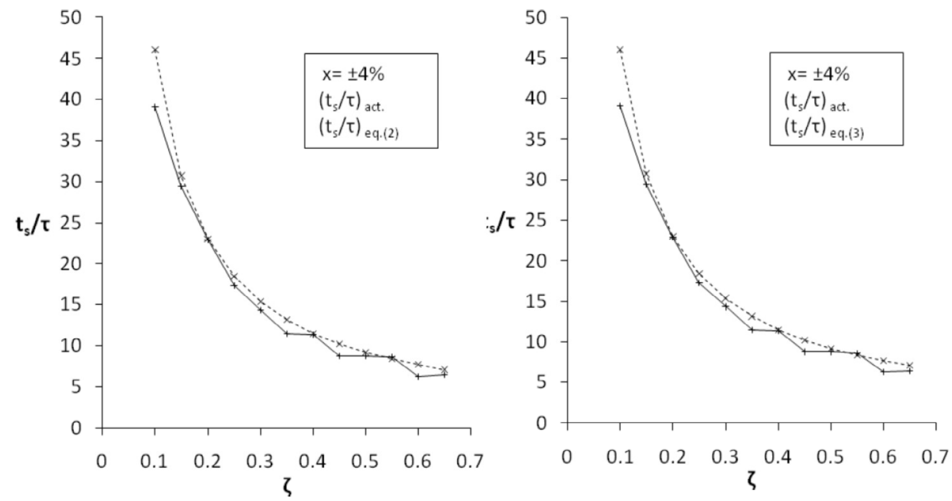

Fig. 4: Actual and calculated (t_s/τ) vs. ζ for ±4% settling band.

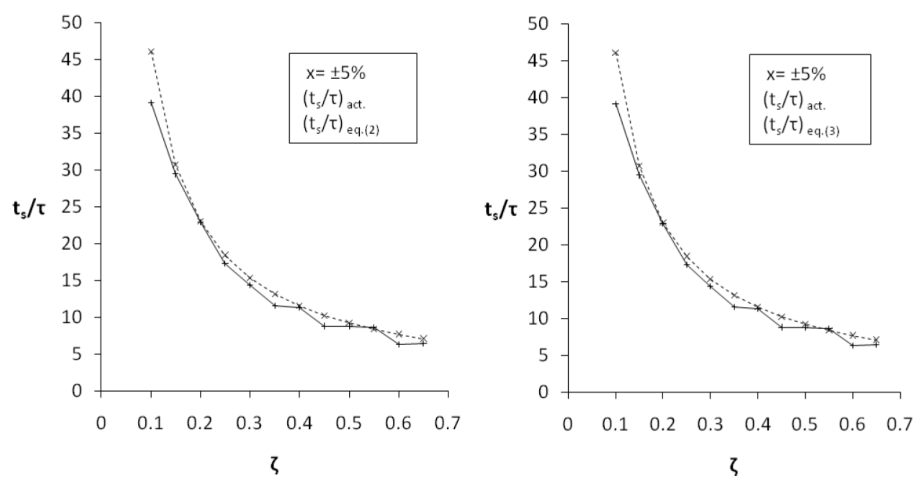

Fig. 5: Actual and calculated (t_s/τ) vs. ζ for ±5% settling band.

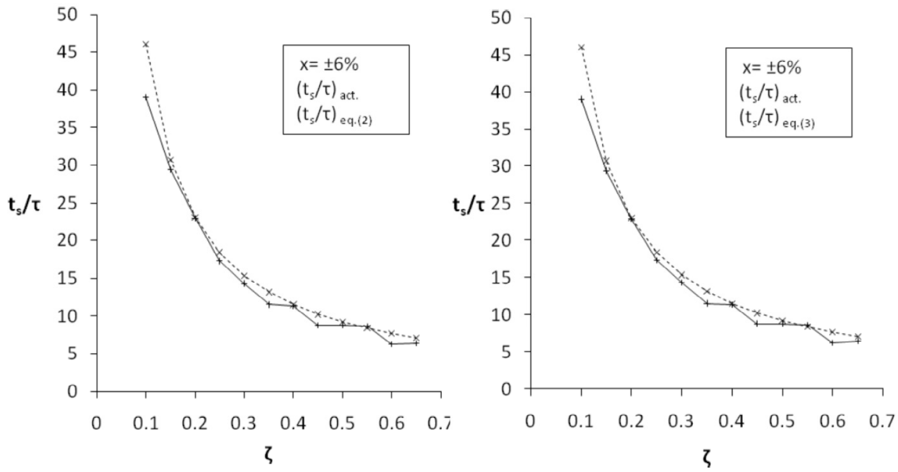

Fig. 6: Actual and calculated (t_s/τ) vs. ζ for ±6% settling band.

3. SETTLING TIME VS. RESPONSE TIME

As a consequence of the adoption of Eq.(1), the following argument is presented as a basis for suggesting that a distinction should be made between t_s and the response time (t_R).

For a 1st-order system subjected to a SCFF, the response time may be defined as that at which the normalized response exceeds 99% of the step change magnitude. Hence, a time interval equal to five times the system's time constant, corresponding to 99.3% of the step change magnitude, would serve this purpose. If this criterion is adopted, then by referring back to Pollard's [2] expression for the LBDE, i.e. $\left(1 - e^{-\frac{\zeta}{\tau}t}\right)$, the normalized response time for an underdamped 2nd-order system would accordingly be $5/\zeta$. In other words, t_R/τ would be the normalized settling time for a ± 0.7% settling band according to Eq.(2), i.e.

$$\frac{t_R}{\tau} = \frac{-\ln(0.007)}{\zeta} = \frac{4.962}{\zeta} \text{ or } \frac{5.0}{\zeta} \qquad (5)$$

Hence, for a ±x% band limits t_s/t_R would be,

$$\frac{t_s}{t_R} = \frac{-\ln(0.01\,x)}{5} \qquad (6)$$

Table (1) gives rounded-off values of t_s/τ, based on Eq.(2), and t_s/t_R, based on Eq.(6), for the range 0.7%≤ ±x ≤6% for comparison purposes.

4. CONCLUSION

A simple mathematical formula is presented to estimate a priori the normalized settling time values of oscillatory 2nd-order systems, when subjected to a SCFF, for any value of the measuring instrument sensitivity. A distinction is made between normalized settling and response times of such systems, with the latter assigned the value of $5/\zeta$. Accordingly, ratios of settling to response times can readily be established.

Table 1 : t_s/τ and t_s/t_R values for 0.7%≤ ±x ≤6%

± x%	ζ (t_s/ τ)	t_s/t_R
0.7	5.00	1.00
1.0	4.60	0.92
1.5	4.20	0.84
2.0	4.00	0.78
2.5	3.70	0.74
3.0	3.50	0.70
3.5	3.35	0.67
4.0	3.22	0.64
4.5	3.10	0.62
5.0	3.00	0.60
5.5	2.90	0.58
6.0	2.81	0.56

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Notation

A SCFF magnitude
K Steady state gain
t_R Response time
t_s Settling time
± x% Sensitivity of measuring instrument, corresponding to settling band limits
Y(t) System's transient response

Greek

ζ Damping coefficient
τ Characteristic time

Abbreviation

act. actual
calc. calculated
eq. equation
LBDE Lower boundary of the decay envelope
SCFF Step change forcing function