On Rare Mutation, Chaos and Darwin’s Theory

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Abstract
According to the Holy Quran, the evolution of life is the result of divine will and divine guidance. One of the key elements of natural selection is that offsprings sometimes vary from their parents. For instance, a yellow butterfly may have a black offspring. This kind of variation and mutation, though rare, tends to occur in all species. Over a long period of time, the cumulative effect of such mutations leads to an entirely new species. Evolution describes the change in genetic material of a population of organisms through successive generations. In this paper, we consider the evolution operator, namely the quadratic stochastic operator which naturally occurs to describe the transmission of a trait from parents to their offsprings. This operator describes evolutionary dynamics of the considered population and its long run behaviour plays an important role in many applied problems. We consider Mendelian inheritance for a single gene with three alleles and assume that prior to the formation of the new generation each gene has the possibility to mutate, that is, to change into a gene of the other kind. We then show that the rare mutation can transform a regular dynamical system into a chaotic one. As a corollary, we have that the evolutionary operator, with rare mutation of alleles, contradict Darwin’s evolution theory on genetic level.

Keywords: Rare Mutation, Regular transformation, Chaos, Darwin’s Theory.

Abstrak

Kata kunci: Sukar bermutasi; Transformasi tetap; Kekeliruan; Teori Darwin.

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Introduction

"Praise be to Allah, Who hath sent to His Servant the Book, and hath allowed therein no Crookedness: (He hath made it) Straight (and Clear) in order that He may warn (the godless) of a terrible Punishment from Him, and that He may give Glad Tidings to the Believers who work righteous deeds, that they shall have a goodly Reward" (Quran 18:1, 2). Evolution as presented by the Holy Quran states that life advanced step by step from dust, water, clay and also from fermenting blackish mud, which subsequently turned into dry, ringing clay (Tahir, 1998) (Quotation based on Holy Quran, Chapter 15, Verses 27, 29 and 34).

According to the Holy Quran, the evolution of life is the result of divine will and divine guidance. The Holy Quran declares that the harmony and complexity of creation and could not have come of its own accord. One of the key elements of natural selection is that offspring sometimes vary from their parents. For instance, a yellow butterfly may have a black offspring. This kind of variation and mutation, though rare, tends to occur in all species. Over a long period of time, the cumulative effect of such mutations leads to an entirely new species. Evolution describes the change in genetic material of a population of organisms through successive generations. The accumulation of these small changes over time can cause substantial changes in a population, a process that can result in the emergence of new species (or speciation). Proponents of this theory postulate that similarities among species suggest that all known species are descended from a common ancestor through this process of gradual divergence. In this paper we consider the evolution operator, namely quadratic stochastic operators, that naturally occurs to describe the transmission of a trait from parents to their offspring. This operator describes evolutionary dynamics of the considered population. The long run behavior of evolution operator plays an important role in many applied problems. In this paper we will explore the discrete nonlinear dynamical system, namely quadratic stochastic operator, that naturally occurs to describe the transmission of a trait from parents to their offspring. We consider Mendelian inheritance for a single gene with three alleles and assuming that prior to the formation of the new generation each gene has the possibility to mutate, that is, to change into a gene of the other kind, and show that the corresponding dynamical system is chaotic. Main goal of this paper is to clarify the role of rare mutation and show that rare mutation of alleles contradict to Darwin’s evolution theory on genetic level.

Definitions

Below we introduce evolutionary operator and justify why it is chosen such kind of operators. Consider a biological population, such as a community of organisms that is closed with respect to reproduction. Assume that each individual in this population belongs to precisely one species \( i = 1, 2, ..., m \). The scale of species is such that the species of the parents \( i \) and \( j \) unambiguously determines the probability of every species \( k \) for the first generation of direct descendants. Denote this probability, that is to be called the heredity coefficient, by \( p_{ij, k} \). It is obvious that \( p_{ij, k} \geq 0 \) and \( \sum_{i=1}^{m} p_{ij, k} = 1 \). Assume that the population is so large that frequency fluctuations can be neglected. Then, the state of the population can be described by the \( m \)-tuple \( (x_1, x_2, ..., x_m) \) of species probabilities, where \( x_k \) is the fraction of the species \( k \) in the total population.

In genetics, random mating involves the mating of individuals regardless of any physical, genetic, or social preference. In other words, the mating between two organisms is not influenced by any environmental, hereditary, or social interaction. Hence, potential mates have an equal chance of being selected. In the case of panmixia (random interbreeding) the parent pairs \( i \) and \( j \) arise for a fixed state \( x = (x_1, x_2, ..., x_m) \) with probability \( x_i x_j \). Hence the total probability of the species \( k \) in the first generation of direct descendents is defined by

\[
x_k = \sum_{i,j=1}^{m} p_{ij, k} x_i x_j
\]

Let

\[
S^{m-1} = \left\{ x = (x_1, x_2, ..., x_m) \in \mathbb{R}^m : x_i \geq 0 \text{ for any } i = 1, ..., m \text{ and } \sum_{i=1}^{m} x_i = 1 \right\}
\]

be the \((m-1)\) dimensional canonical simplex in \(\mathbb{R}^m\).

Definition 2.1 The transformation \( V : S^{m-1} \rightarrow S^{m-1} \) is called a quadratic stochastic operator if

\[
(2.1) \quad (Vx)_k = \sum_{i,j=1}^{m} p_{ij, k} x_i x_j, \quad k = 1, ..., m
\]

where a) \( p_{ij, k} \geq 0 \); b) \( p_{ij, k} = p_{ji, k} \); and c) \( \sum_{i=1}^{m} p_{ij, k} = 1 \) for arbitrary \( i, j, k = 1, 2, ..., m \).
We introduce some standard terms in the theory of a discrete dynamical system $V : S^{m-1} \to S^{m-1}$. A sequence $\{x^{(n)}\}_{n=0}^{\infty}$ is called a trajectory of $V$ starting from an initial point $x^0$, where $x^{(n)} = V(x^{(n-1)})$ for any $n \in \mathbb{N}$. Recall that a point $x$ is called a fixed point of $V$ if $V(x) = x$. We denote a set of all fixed points by $\text{Fix}(V)$.

**Definition 2.2** A dynamical system $V$ is called regular if a trajectory $\{x^{(n)}\}_{n=0}^{\infty}$ converges for any initial point $x$.

Note that if $V$ is regular, then the limiting point is a fixed point of $V$. Thus, in a regular system, the fixed point of dynamical system $V$ describes a long run behavior of the trajectory of $V$ starting from any initial point. The biological treatment of the regularity of dynamical system $V$ is rather clear: in a long run time, the distribution of species in the next generation coincide with distribution of species in the current generation, i.e., stable. A fixed point set and an omega limiting set of quadratic stochastic operators (QSO) were deeply studied in (Kesten, 1970; Ganikhodzhaev, 1993; Ganikhodzhaev, 1994; Ganikhodzhaev and Eshmamatova, 2006; Jenks, 1969), and they play an important role in many applied problems (Bernstein, 1924; Lyubich, 1992). In the paper (Ganikhodzhaev et al., 2011), it was given a long self-contained exposition of recent achievements and open problems in the theory of quadratic stochastic operators. Note that the limit point is a fixed point of a quadratic stochastic operator $V$. Thus, the fixed points of a quadratic stochastic operator describe limit or long run behavior the trajectories of any initial points.

**Definition 2.3** A quadratic stochastic operator $V : S^{m-1} \to S^{m-1}$ is said to be ergodic if the limit

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} V^k(x)$$

exists for any $x \in S^{m-1}$.

Based on some numerical calculations, S. Ulam has conjectured (Ulam, 1960) that any QSO $V : S^{m-1} \to S^{m-1}$ acting on the finite dimensional simplex is ergodic. However, in (Zakharevich, 1978), M. Zakharevich showed that, in general, Ulam’s conjecture is false. Namely, M. Zakharevich showed that the following QSO $V_0 : S^2 \to S^2$ is not ergodic

$$V_0 : \begin{cases} x'_1 = x_1^2 + 2x_1x_2 \\ x'_2 = x_2^2 + 2x_2x_3 \\ x'_3 = x_3^2 + 2x_3x_1 \end{cases}$$

In (Ganikhodzjaev and Zanin, 2004; Saburov 2012; Saburov, 2012a and Saburov, 2013), Zakharevich’s result was generalized in the class of Volterra QSO. Moreover, in (Ganikhodzjaev and Zanin 2004), it was given a necessary and sufficient condition of being non-ergodicity of Volterra QSO defined on $S^2$. Note that if QSO is regular, then it is ergodic. However, the reverse implication is not always true.

Recently chaotic dynamical systems are very popular in science and engineering. Besides the original definition of Li-Yorke chaos in (Li and Yorke, 1975), there have been various definitions for "chaos" in the literature, and the most often used one is given by Devaney in (Devaney, 1989). Although there is no universal definition for chaos, the essential feature of chaos is sensitive dependence on initial conditions so that the eventual behavior of the dynamics is unpredictable. The theory and methods of chaotic dynamical systems have been of fundamental importance not only in mathematical sciences, but also in physical, engineering, biological, and even economic sciences. In this paper, a chaos would be understood in the sense of Li-Yorke (see Blanchard et al., 2002). In this paper, we introduce and examine a family of nonlinear discrete dynamical systems that naturally occur to describe a transmission of a trait from parents to their offspring. Here, we shall present some essential analytic and numerical results on dynamics of such models.

**Inheritance for a Single Gene with Two Alleles**

As the first example, we consider a Mendelian inheritance of a single gene with two alleles $A$ and $a$ (see Reed, 1997). Let an element $x$ represent a gene pool for a population and it have been expressed a linear combination of the alleles $A$ and $a$

$$x = x_A A + x_a a$$

where, $0 \leq x_A, x_a \leq 1$ and $x_A + x_a = 1$. Then, $x_A, x_a$ are the percentage of the population which carries the alleles $A$ and $a$ respectively. The rules of the Mendelian inheritance indicate that the next generation

$$x' = x_A A + x_a a$$

represents a gene pool for the population which carries the alleles $A$ and $a$ respectively, where
Here, $P_{AA,A}$ (resp. $P_{AA,a}$) is the probability that the child receives the allele $A$ (resp. $a$) from parents with the allele $A$; $P_{aA,A}$ (resp. $P_{aA,a}$) is the probability that the child receives the allele $A$ (resp. $a$) from parents with the alleles $A$ and $a$ respectively; and $P_{aa,A}$ (resp. $P_{aa,a}$) is the probability that the child receives the allele $A$ (resp. $a$) from parents with allele $a$. It is evident that $P_{AA,A} + P_{aA,A} = 1$, $P_{aA,A} = P_{aA,a}$, $P_{AA,a} = P_{aa,a}$, with $x_1 + x_2 = 1$.

Thus, (3.1) is a nonlinear dynamical system acting on the one-dimensional simplex

$$S^1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0, x_1 + x_2 = 1\}$$

that describes the distribution of the next generation which carries the alleles $A$ and $a$ respectively, if the distribution of the current generation are known. Recall that in the Mendelian inheritance case, i.e., $P_{AA,A} = P_{aA,a} = 1$ and $P_{AA,a} = P_{aa,A} = 0$, the dynamical system (3.1) has the following form

$$\begin{cases}
  x_1' = P_{AA,A}x_1^2 + 2P_{aA,A}x_1x_2 + P_{aa,A}x_2^2 \\
  x_2' = P_{aa,a}x_1^2 + 2P_{aA,a}x_1x_2 + P_{AA,a}x_2^2
\end{cases}$$

(3.1)

We assume that prior to a formation of a new generation each gene has a possibility to mutate, that is, to change into a gene of the other kind. Specifically, we assume that for each gene the mutation $A \rightarrow a$ occurs with probability $\alpha$, and $a \rightarrow A$ occurs with probability $\beta$. We assume that "the mutation occurs if only if both parents have the same allele". Then, we have that $P_{AA,a} = \alpha$, $P_{aA,a} = \beta$, $P_{AA,A} = 1 - \alpha$, $P_{aa,a} = 1 - \beta$, and the dynamical system (3.2) has the following form

$$\begin{cases}
  x_1' = (1 - \alpha)x_1^2 + 2P_{aA,a}x_1x_2 + \beta x_2^2 \\
  x_2' = \alpha x_1^2 + 2P_{aA,a}x_1x_2 + (1 - \beta)x_2^2
\end{cases}$$

(3.2)

Thus the operator $V : S^1 \rightarrow S^1$ given by (3.3) is a quadratic stochastic operator. It is known that the dynamical system (3.3) is either regular or converges to a periodic-2 point (see Ljubic, 1978). Therefore, in 1D simplex, any QSO is ergodic. In other words, a mutation in population system having a single gene with two alleles always exhibits an ergodic behavior (or almost regular). It is of independent interest to study a mutation in population system having a single gene with three alleles $a_1, a_2$ and $a_3$ and show that a nonlinear dynamical system corresponding to the mutation exhibits a non-ergodic behavior (or Li-Yorke chaos).

### Inheritance for a Single Gene with Three Alleles

In this section, we shall derive a mathematical model of an inheritance of a single gene with three alleles. In this case, an element $x$ represents a population if its expression $x = x_a a_1 + x_a a_2 + x_a a_3$, as a linear combination of the alleles $a_1, a_2$ and $a_3$, satisfies the following conditions $0 \leq x_1, x_2, x_3 \leq 1$ and $x_1 + x_2 + x_3 = 1$. Then $x_1, x_2, x_3$ are the percentage of the population which carries the alleles $a_1, a_2$ and $a_3$ respectively.

We assume that prior to a formation of a new generation each gene has a possibility to mutate, that is, to change into a gene of the other kind. We assume that the mutation occurs if both parents have the same alleles. Specifically, we will consider two types of the simplest mutations:

1. Assume that mutations $a_1 \rightarrow a_2$, $a_2 \rightarrow a_3$, and $a_3 \rightarrow a_1$ occur with probability $\alpha$;
2. Assume that mutations $a_1 \rightarrow a_3$, $a_2 \rightarrow a_1$, and $a_3 \rightarrow a_2$ occur with probability $\alpha$.

Then the corresponding dynamical systems are defined on the two-dimensional simplex

$$S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x \geq 0, x_2 \geq 0, x_3 \geq 0, x_1 + x_2 + x_3 = 1\}.$$

In the first mutation, we have (4.1)

$$\begin{cases}
  x_1' = (1 - \alpha)x_1^2 + 2x_1x_2 + \alpha x_2^2 \\
  V_\alpha : x_2' = (1 - \alpha)x_2^2 + 2x_2x_3 + \alpha x_3^2, \\
  x_3' = (1 - \alpha)x_3^2 + 2x_3x_1 + \alpha x_1^2
\end{cases}$$

In the second mutation, we have (4.2)

$$\begin{cases}
  x_1' = (1 - \alpha)x_1^2 + 2x_1x_2 + \alpha x_2^2 \\
  W_\alpha : x_2' = (1 - \alpha)x_2^2 + 2x_2x_3 + \alpha x_3^2, \\
  x_3' = (1 - \alpha)x_3^2 + 2x_3x_1 + \alpha x_1^2
\end{cases}$$
In both cases, if $\alpha = 0$, i.e., if a mutation does not occur, then a dynamical system (4.1) and (4.2) coincide with Zakharevich's operator (2.3). In next section we consider attractors of dynamical systems (4.1) and (4.2) for a rare mutation, i.e., rather small $\alpha$.

**From Order to Chaos by Rare Mutations**

*Chaos in the first type mutation*

We are going to present some pictures of attractors (an omega limiting set) of the operator $V_\alpha : S^2 \to S^2$ given by (4.1).

In the cases $\alpha = 0$ and $\alpha = 1$, the operators $V_0$ and $V_1$ have similar behaviors. The trajectories of both operators, $V_0$ and $V_1$, look as spirals, but one of them is moving by clockwise and another one is moving by anticlockwise. In these cases, we have that $\omega_{V_0}(x^0) \subset \partial S^2$ and $\omega_{V_1}(x^0) \subset \partial S^2$. We are interested in the dynamics of the mutation operator $V_\alpha$ while $\alpha$ approaches to $\frac{1}{2}$ from both left and right sides. In order to see some symmetry, we shall provide attractors of $V_\alpha$ and $V_{1-\alpha}$ at the same time.

If we slightly change $\alpha$ from 0 and 1, then we can see that the omega limiting set splits from the boundary. Moreover, in the picture, we can see "three vertexes" (it is roughly speaking) where the trajectory shall spend almost all its time around those points. Therefore, we are expecting non-ergodicity of the operator $V_\alpha$ if $\alpha$ is near by 0 or 1.

If $\alpha$ becomes close to the $\frac{1}{2}$ then we can see some chaotic pictures. We observe from the pictures that, in the cases $\alpha$ and $1 - \alpha$, the attractors are the same but different from the orientations. One of them is moving by clockwise and another one is by anticlockwise. If $\alpha$ is enough close to $\frac{1}{2}$ then we detect completely different pictures. Here are some pictures for the values of $\alpha = 0.4995, 0.4999, 0.5005, 0.5001$. In this cases, attractors are disconnected sets and they consist of 6 connected sets. It is very surprising that why the number of connected sets are 6 and why not 5 or 7.

For the operator (4.1), the fluctuation point is $\alpha = \frac{1}{2}$. In this case, the influence of the chaotic operators $V_0$ and $V_1$ are the same. Therefore, the operator $V_{0.5}$ becomes regular and ergodic. This completes the numerical study of the operator (4.1).
**Chaos in the second type mutation**

We are going to present some pictures of attractors (an omega limiting set) of the operator $W_\alpha : S^2 \to S^3$ given by (4.2). In this cases for $\alpha = 0$ the operator $W_0$ is a chaotic operator and for $\alpha = 1$, the operator $W_1$ is regular. Since $W_\alpha = (1 - \alpha)W_0 + \alpha W_1$, the mutation operator $W_\alpha$ gives a transition from the chaotic behavior to the regular behavior. Consequently, we are aiming to find fluctuation points in which we can see the transition from the chaotic behavior to the regular behavior.

If $\alpha$ is very close to 0 then attractors of the operator $W_\alpha$ are splitted from the boundary of the simplex. However, the influence of the operator $W_0$ is still very high. Therefore, we are expecting that the operator $W_\alpha$ is non-ergodic and chaotic, whenever $\alpha$ is very close to 0. If $\alpha$ becomes to close to 0.13333 then we can see a different picture. In attractors, we are able to detect some "growing hairs" (it is a just terminology). Moreover, if we shall continue to increasing $\alpha$ then these "hairs" will start to rise and they eventually become straight. Therefore, $0.13333 = \alpha$ is the first fluctuation point. It is very interesting that the number of "hairs" is equal to 12. It is again a question: why do we have 12 numbers of "hairs" and why not 11 or 13.

From these pictures, we can find another fluctuation point $\alpha = 1 - \frac{\sqrt{3}}{2} \cdot 0.1339745$.

Therefore, in order to have a transition from a chaotic behavior to the regular behavior, we should cross from two fluctuation points $\alpha = 0.13333$ and $\alpha = 1 - \frac{\sqrt{3}}{2} \approx 0.1339745$.

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**Figure 3**: Attractors of $V_\alpha$: $\alpha = 0.4995$ and $\alpha = 0.5005$

**Figure 4**: Attractors of $V_\alpha$: $\alpha = 0.4999$ and $\alpha = 0.5001$
Figure 6: Attractors of $W_{\alpha}$: $\alpha = 0.001$, $\alpha = 0.01$ and $\alpha = 0.1$

Figure 7: Attractors of $W_{\alpha}$: $\alpha = 0.13333$ and $\alpha = 0.1338$

Figure 8: Attractors of $W_{\alpha}$: $\alpha = 0.139$, $\alpha = 0.144$, and $\alpha = 0.150$
Conclusion
In this paper, we present a mathematical model of the mutation in the biological environment having 3 alleles. We have presented two types of mutations. We have shown that a mutation (a mixing) in the system can be considered as a convex combination of Mendelian inheritances (extreme or non-mixing systems). The first mutation is a convex combination of two Li-Yorke chaotic systems and the second mutation is a convex combination of Li-Yorke chaotic and regular systems. In the first case, the first mutation can be considered as an evolution process between two different chaotic biological systems.

In this case, we have shown that there is one fluctuation point (transition point) in order to change one chaotic system into another chaotic system. In the second case, the second mutation can be considered as an evolution process between chaotic and regular biological systems. In this case, we need two fluctuation points (transition points) in order to change the chaotic system into the regular system.

The existence of fluctuation points contradicts to Darwin’s evolution theory on genetic level, since under rare mutation of alleles the evolutionary operator can change regular behavior into chaotic behavior.

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